and taking square roots yields the CSB $\leq$. ■

Inner product on $\mathbb{C}^n$ We have to give up something, and experience shows it's best to define a complex inner product so that $\langle v,v\rangle$ is the length of v when it is viewed as a vector in $\mathbb{R}^{2n}$:

$$\text{complex } \langle v,w\rangle = \sum_{j=1}^{n} v_j \overline{w_j} \leftarrow \text{conjugation}$$

If $v = x+iy$ then $\langle v,v\rangle$ becomes

$$\sum v_j \overline{v_j} = \sum x_j^2 + y_j^2, \text{ as desired.}$$

This leads to a modified commutative law:

$$\langle w,v\rangle = \overline{\langle v,w\rangle}$$

and also a modified homogeneity property:

$$c\langle v,w\rangle = \langle cv,w\rangle = \langle v,\overline{c}w\rangle \text{ or}$$

$$\text{equivalently, } \langle v,cw\rangle = \overline{c}\langle v,w\rangle.$$

<u>Abstraction</u> An <u>inner product space</u> over $\mathbb{R}$ or $\mathbb{C}$ is a vsp V plus an inner product $\langle , \rangle : V \times V \longrightarrow \mathbb{C}$ which satisfies the given properties.

See Axler, p. 166 for examples.

<u>Still more</u>: If $d_i \neq 0$ for $i = 1, \ldots, n$ then $\sum |d_j|^2 v_j \bar{w}_j$ defines an inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$ ($\bar{a} = a$ if a is real).

Note that the length $|v| = \sqrt{\langle v, v \rangle}$ of a vector in an innerproduct space will satisfy $|v| \geq 0$, with $|v| = 0 \Leftrightarrow v = 0$
$$|cv| = |c| \cdot |v|.$$

<u>MORE GEOMETRY.</u>

Two vectors are $\left\{ \begin{matrix} \text{perpendicular} \\ \text{orthogonal} \end{matrix} \right\}$ if $\langle v, w \rangle = 0$. $(\Leftrightarrow \langle w, v \rangle = 0)$

In fact, the Cauchy-Schwarz-Buniakovsky inequality extends to all inner product spaces!

As before, need only consider the case $a, b \neq 0$.

<u>Derivation</u>

Write $x = b - \dfrac{\langle b,a\rangle}{\langle a,a\rangle} a$. Then

$$\langle x,a\rangle = \langle b,a\rangle - \frac{\langle b,a\rangle}{\langle a,a\rangle}\langle a,a\rangle = 0,$$

so

$$|b|^2 = \left\langle x + \frac{\langle b,a\rangle}{\langle a,a\rangle} a,\ x + \frac{\langle b,a\rangle}{\langle a,a\rangle} a \right\rangle =$$

$$|x|^2 + \frac{\langle b,a\rangle}{\langle a,a\rangle}\underbrace{\langle x,a\rangle}_{0} + \frac{\overline{\langle b,a\rangle}}{\langle a,a\rangle}\underbrace{\langle x,a\rangle}_{0} + \frac{|\langle a,b\rangle|^2}{|a|^4}|a|^2$$

Hence $0 \le |x|^2 = |b|^2 - \dfrac{|\langle a,b\rangle|^2}{|a|^2}$ or

$|b|^2 \ge \dfrac{|\langle a,b\rangle|^2}{|a|^4}$, which means

$|a|^2|b|^2 \ge |\langle a,b\rangle|^2$. Take sq roots to get the desired inequality.

Suppose equality holds. Then $|x|^2$ must be 0, which means b is a scalar multiple of a.

Conversely, $b = k\cdot a \Rightarrow |\langle a, \underset{\overset{\shortparallel}{ka}}{b}\rangle|^2 = |k|^2|a|^2 =$

$|a|^2|ka|^2 = |a|^2|b|^2$. ■

Here is an even more geometrically motivated result: