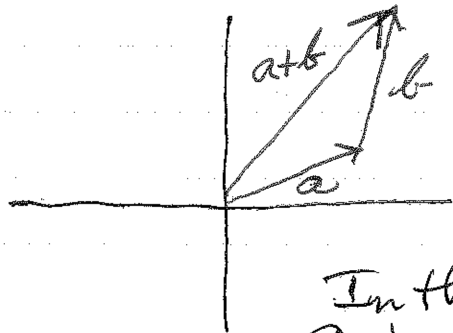
TRIANGLE INEQUALITY



Over $\mathbb{R}$ or $\mathbb{C}$,
$|a+b| \le |a| + |b|$.

In this picture strict $\le$ holds.
But $=$ holds if, say, $a=b$.

<u>Derivation</u> Need only prove squares are unequal in the same order.

$$|a+b|^2 = \langle a+b, a+b\rangle = |a|^2 + \langle a,b\rangle + \langle b,a\rangle + |b|^2 =$$

(since $\langle b,a\rangle = \overline{\langle a,b\rangle}$) $|a|^2 + 2\,\mathrm{real}\langle a,b\rangle + |b|^2 =$

(since real part $x+yi \le |x+yi|$) $|a|^2 + 2|\langle a,b\rangle| + |b|^2 \le$

(C-S-B inequality) $|a|^2 + 2|a||b| + |b|^2 =$

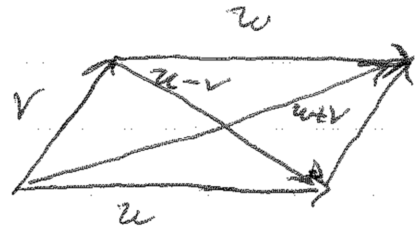
$$(|a| + |b|)^2.$$

To finish, take sq roots of the first + last expressions. ∎

If a + b are linearly independent, then strict inequality holds because $|\langle a,b\rangle| <$ $|a| \cdot |b|$ in such cases.

One more result:

Parallelogram Law



Over $\mathbb{R}$,

$$|u+v|^2 + |u-v|^2 = 2\left(|u|^2 + |v|^2\right)$$

Derivation The left hand side is

$$|u|^2 + \cancel{2\langle u,v\rangle} + |v|^2 + |u|^2 - \cancel{2\langle u,v\rangle} + |v|^2$$

$$= 2|u|^2 + 2|v|^2. \blacksquare$$

See Axler, p. 174, for a proof that works in the complex case.