6B. Orthogonality and dimension

One way of characterizing 2- and 3-dimensional vectorspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$ is that there are 2 or 3 (resp) distinct and mutually perpendicular directions. One aim of this section is to generalize this intuitive notion to higher dimensions.

<u>Prop.</u> If $\{v_1, \ldots, v_m\}$ is a set of nonzero, mutually perpendicular vectors in an inner product space V, then $\{v_1, \ldots, v_m\}$ is linearly independent.

<u>PROOF.</u> Suppose $\sum c_j v_j = 0$. For each k, we then have $0 = \langle \sum c_j v_j, v_k \rangle = \sum c_j \langle v_j, v_k \rangle$. Since $\langle v_j, v_k \rangle = 0$ if $j \neq k$, this yields $c_k |v_k|^2 = 0$, and since $v_k \neq 0$ we have $|v_k|^2 > 0$ and hence $c_k = 0$. Therefore the given set of vectors is lin. indep. ■

Hence $\dim V = n \Rightarrow$ every set of nonzero mutually $\perp$ vectors has $\leq n$ elements.

<u>Question</u> Is there such a set with exactly $n = \dim V$ elements?

<u>Example</u> $V = \mathbb{R}^n$ or $\mathbb{C}^n$, e_j = unit vector with jth coord = 1, all others equal to zero. Then $\{e_1, \dots, e_n\}$ is mutually orthogonal.

<u>Orthonormal set.</u> $\{v_1, v_2, \dots, v_n\}$ is orthonormal if the vectors are mutually $\perp$ and $|v_j| = 1$ for all j.

Gram-Schmidt Orthonormalization Process

Let V be an ~~vector~~ inner product space with basis $\{v_1, \dots, v_n\}$. Then V has an orthonormal basis $\{u_1, \dots, u_n\}$ such that for each k we have

$$\operatorname{Span}\{u_1, \dots, u_k\} = \operatorname{Span}\{v_1, \dots, v_k\}.$$

<u>PROOF</u> Note that

(1) This applies to fin dim subspaces of inner product spaces (restrict the inner product).

(2) The conclusion suggest that $\{u_1, \dots, u_n\}$ is constructed recursively, and it is.