<u>$k=1$</u> $v_1 \neq 0$; let $u_1 = \frac{1}{|v_1|} v_1$.

<u>Suppose true for $k<n$, $k \geq 1$</u>

span same subspace

Find a vector $w \perp \{v_1, \ldots, v_k\}, \{u_1, \ldots, u_k\}$

such that w is a lin comb. of $v_1, v_2, \ldots, v_k, v_{k+1}$:

$$W = V_{k+1} - \sum_{j=1}^{k} \langle w, u_j \rangle u_j .$$

This vector is $\perp u_1, \ldots, u_k$. Now if w is perpendicular to $a_1, \ldots, a_k$ then it is also $\perp$ to every linear combination of $a_1, \ldots, a_k$ because

$$\langle w, \sum y_j a_j \rangle = \sum \overline{y_j} \underbrace{\langle w, a_j \rangle}_{=0} = 0 .$$

Since nonzero orthogonal sets are linearly indep., w isn't a lin. comb. of $v_1, \ldots, v_k$ (otherwise v_{k+1} would be). In particular, $w \perp u_j$ and $w \neq 0$ & $\mathrm{Span}\{u_1, \ldots, u_k, w\} = \mathrm{Span}\{v_1, \ldots, v_{k+1}\}$.

Finally, let $u_{k+1} = \frac{1}{|w|} w$; then $\{u_1, \ldots, u_{k+1}\}$ is orthonormal with the same span as $\{v_1, \ldots, v_{k+1}\}$.

The G.-S. process provides an effective algorithm for converting an arbitrary basis of $\mathbb{R}^n$ to an orthonormal one, and likewise for all subspaces of $\mathbb{R}^n$. It would have been nice if the text book had given a worked example. See <u>gram-schmidt.pdf</u> for an example from Schaum's Outline of Linear Algebra.

<u>A FEW USEFUL IDENTITIES</u>

Let $\{u_1, \dots, u_n\}$ be an orthonormal basis for V.

① If $x \in V$ and $x = \sum c_j u_j$, then

$$c_j = \langle x, u_j \rangle \quad \text{(Formula 6.30, p 182)}$$

② If $x = \sum a_j u_j$ and $y = \sum b_j u_j$, then

$$\langle x, y \rangle = \sum a_j \bar{b}_j .$$

Also look at Thm. 6.38 (p.186), Example 6.29 (p.181), Thm. 6.35 (p.185)