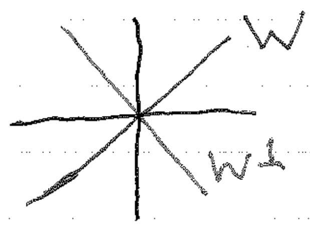
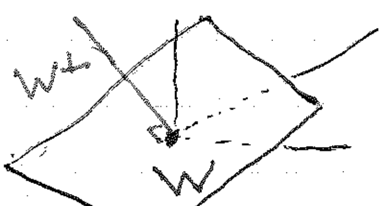
6.C Orthogonal complements and least squares

V = inner product space, $W \subseteq V$ subspace.
The orthogonal complement $W^\perp$ = all $x \in V$ such that $\langle x, w\rangle = 0$ for all $w \in W$.



$\mathbb{R}^2$



in $\mathbb{R}^3$

Simple facts.

$W^\perp$ is a subspace

$x, y \in W^\perp$ c scalar $\Longrightarrow$

$\langle x+y, w\rangle = \langle x, w\rangle + \langle y, w\rangle = 0 + 0 = 0$ all $w \in W$

$\langle cx, w\rangle = c\langle x, w\rangle = c \cdot 0 = 0$ all w.

$\{0\}^\perp = V$, $V^\perp = \{0\}$, $W \cap W^\perp = \{0\}$ all W

$W_1 \subseteq W_2 \Longrightarrow W_2^\perp \subseteq W_1^\perp$.

<u>Theorem</u> If V is finite-dimensional, then
$$W + W^{\perp} = V.$$

<u>Corollary</u> Every $x \in V$ has a unique decomp. $x_1 + x_2$ where $x_1 \in W$, $x_2 \in W^{\perp}$.

<u>Proof of Corollary</u> Existence of a decomposition is the conclusion of the theorem. For uniqueness, say $x_1 + x_2 = y_1 + y_2$ $x_1, y_1 \in W$, $x_2, y_2 \in W^{\perp}$

Then $\underbrace{x_1 - y_1}_{\text{in } W} = \underbrace{y_2 - x_2}_{\text{in } W^{\perp}}$. Since $W \cap W^{\perp} = \{0\}$,

we must have $x_1 - y_1 = 0 = y_2 - x_2$, so that $x_1 = y_1$ and $x_2 = y_2$. ∎

<u>Proof of thm.</u> Let $\{w_1, \ldots, w_r\}$ be an orthonormal basis for W. If $x \in V$, then

$$x_2 = x - \underbrace{\sum \langle x, w_j \rangle w_j}_{= x_1} \in W^{\perp} \text{ by previous calculation.}$$

Thus $x_2 = x - x_1 \Rightarrow x = x_1 + x_2$, $x_1 \in W$, $x_2 \in W^{\perp}$. ∎

<u>Cor.</u> $\dim V = \dim W + \dim W^{\perp}$, or equivalently $\dim W^{\perp} = \dim V - \dim W$.

PROOF. $W + W^{\perp} = V$, $W \cap W^{\perp} = \{0\} \Longrightarrow$

$$\dim W + \dim W^{\perp} = \dim(W + W^{\perp}) + \dim(W \cap W^{\perp})$$
$$= \dim V + 0. \blacksquare$$

Corollary $\dim V$ finite $\Longrightarrow$

$$W = (W^{\perp})^{\perp}.$$

Proof. $\dim W^{\perp\perp} = \dim W - \dim W^{\perp} = \dim W$.

On the other hand

$x \in W \Longrightarrow \langle x, y \rangle = 0$ all $y \in W^{\perp} \Longrightarrow W^{\perp\perp}$.
Hence $W \subseteq W^{\perp}$. If $r = \dim W = \dim W^{\perp\perp}$, then if $\{w_1, \dots, w_r\}$ is a basis for W, it must also be a basis for $W^{\perp\perp}$, which has the same dimension. Hence $W \supseteq W^{\perp\perp}$, so $W = W^{\perp\perp}$. $\blacksquare$

Orthogonal Projection onto W

V = finite dimensional inner product space, $W \subseteq V$ subspae. The orthogonal (perpendicular) projection $E_W(v)$ of v onto W is defined by x_1, where $v = x_1 + x_2$ where $x_1 \in W$, $x_2 \in W^{\perp}$.