7A. Linear transformations and inner products

DEFAULT HYPOTHESIS. All vector spaces are finite dimensional.

Theorem ("Riesz Representation" in Axler; actually well known much earlier in the 19th century.

Notation If V is a vector space, a *linear functional* is a linear transformation $V \longrightarrow \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$).

If V is equipped with an inner product $\langle , \rangle$ and $f : V \longrightarrow \mathbb{F}$ is a linear functional, then there is a unique $x_f \in V$ such that for all $y \in V$ we have $f(y) = \cancel{\langle x_f, y \rangle}\ \langle y, x_f \rangle$.

Derivation Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis for V.

Existence. Suppose that $f(u_j) = c_j$. Let

$x_f = \sum \overline{c_j} u_j$. note the conjugation. Then $f(u_j) = \langle u_j, x_f \rangle$

all j, and if $y = \sum a_k u_k$, then

$f(y) = \sum a_k f(u_k) = \sum a_k c_k =$

$\langle \sum a_k w_k, \sum \bar{c}_j w_j \rangle = \langle y, x_f \rangle$. ■

Uniqueness. Suppose $\langle y, x \rangle = \langle y, x' \rangle$ for all $y \in V$. Then $\langle y, x - x' \rangle = 0$ all y, so $x - x' \in V^\perp = \{0\}$. ■

Adjoint transformations

<u>Theorem</u>. Let $T : V \to W$ be a linear transf. of inner product spaces. Then there is a unique linear transformation $T^* :$ ~~$W^* \to V^*$~~ $W \to V$ such that

$$\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V .$$

This map T^* is called the <u>adjoint transformation</u>.

<u>Proof.</u> Existence Notice that if $w \in W$ then $f(v) = \langle Tv, w \rangle$ is a linear functional on V. Therefore $\langle Tv, w \rangle = \langle v, z \rangle$ for some unique $z \in V$; set $T^* w = v$.

Need to show T^* is linear: $\langle v, T^*(w_1 + w_2) \rangle =$
$\langle Tv, w_1 + w_2 \rangle = \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle =$

$\langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle = \langle v, T^* w_1 + T^* w_2 \rangle$ for all $v \in V$. By the previous result, $T^*(w_1 + w_2) = T^* w_1 + T^* w_2$. Also

$\langle v, T^* cw \rangle = \langle Tv, cw \rangle = \bar{c} \langle Tv, w \rangle = \bar{c} \langle v, T^* w \rangle = \langle v, \bar{c} T^* w \rangle$, and as before $T^*(cw) = cT^* w$. ■

Formal properties $(S+T)^* = S^* + T^*$

$(cT)^* = \bar{c} T^*$

$T^{**} = T \qquad I^* = I$

$(ST)^* = T^* S^*$.

Derivations are on Axler, p. 206

Matrix representation Let $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_m\}$ be orthonormal bases for V & W respectively, and let $T: V \to W$ be linear.

$\boxed{\text{CLAIM}}$ If $Tv_j = \sum a_{ij} w_i$, then $\langle Tv_j, w_i \rangle = a_{ij}$.

Then $T^* w_j = \sum b_{ij} v_i \Longrightarrow$

$$b_{ij} = \langle T^* w_j, v_i \rangle = \overline{\langle v_i, T^* w_j \rangle} = \overline{\langle T v_i, w_j \rangle} = \overline{a_{ji}}$$

conjugate transpose.

(ordinary transpose over $\mathbb{R}$)

More identities

$$\text{Kernel } T^* = (\text{Image } T)^\perp$$
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Note that the first two and $T^{**} = T$ imply the last two.

See Axler, p. 207, for derivations.