# Adjoints and diagonalization

Goal: Show that if $T$ and $T^*: V \to V$ are closely related (for example, $T = T^*$), then $T$ has an orthonormal basis of eigenvectors.

Def. $T: V \to V$ is self-adjoint if $T = T^*$.

In terms of matrices, if $T$ is rep. by $A$, then $T^*$ is rep by the $\boxed{\text{conjugate transpose}}$ ${}^T\bar{A} = A^*$. (${}^TA = A$ transposed).

If $A$ is a matrix with real entries, $A^* = A$ means ${}^TA = A$, or $A$ is symmetric.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{typical example.}$$

Over $\mathbb{C}$, we say $A$ is Hermitian if $A^* = A$
(her-MEE-shen)
(after C. Hermite, 1822-1901)

$$\text{typical example} \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

<u>PROPOSITION.</u> If $T^* = T$, then all eigenvalues are real.

<u>Proof</u> Say $Tv = cv$, $v \neq 0$. Then

$$c \cdot |v|^2 = c\langle v,v\rangle = \langle cv, v\rangle = \langle Tv, v\rangle$$

$$= \langle v, T^*v\rangle = \langle v, cv\rangle = \bar{c}\langle v,v\rangle = \bar{c}\,|v|^2.$$

Since $|v|^2 > 0$, we can cancel it and conclude that $c = \bar{c}$, so that $c$ is real. ∎

<u>THEOREM. (Principal Axis Theorem)</u> If $A$ is <u>real</u> symmetric, then $A$ has a real eigenvalue.

<u>Proof</u> If $A$ is invertible over $\mathbb{R}$, then $A$ is invertible over $\mathbb{C}$. Taking contrapositives, if $A$ is not invertible over $\mathbb{C}$ and $A$ is real, then $A$ is not invertible over $\mathbb{R}$.

${}^T\!A = A$ and the previous result imply $A - cI$ is not $\mathbb{C}$ invertible for some $c \in \mathbb{R}$, so it also isn't $\mathbb{R}$ invertible.

Therefore $A$ must have a real eigenvector for $c \in \mathbb{R}$.

What follows is out of order from Axler.

<u>Theorem (Diagonalization)</u> If $A^* = A$ over $\mathbb{R}$ or $\mathbb{C}$, then $A$ has an <u>orthonormal</u> basis of eigenvectors over $\mathbb{R}$ or $\mathbb{C}$. → suffices to find an orthogonal basis.

In particular, real-symmetric matrices are diagonalizable.

<u>Proof</u> Assume $T^* = T$ throughout this proof, which works over $\mathbb{R}$ or $\mathbb{C}$. Prove the result for $T : V \to V$ (inner product space) with $T^* = T$.

<u>$\dim V = 1$</u> True since $0 \neq v \Rightarrow Tv = cv$.

Assume if $\dim V = k-1$, where $k > 1$.

<u>Inductive step</u> Know $Tv = cv$ some $v \neq 0$, scalar $c$. Claim: If $W = \operatorname{Span}(\{v\})^{\perp}$, then

$T[W] \subseteq W$ (and $T^*[W] \subseteq W$).

Since $T = T^*$, we know $T^* v = cv$ also holds. Let $x \in W$, so that $\langle v, x \rangle = 0$. But then $\langle v, Tx \rangle = \langle Tv, x \rangle = \langle cv, x \rangle = c\langle v, x \rangle = c \cdot 0 = 0$. If we let $S: W \to W$ be the associated linear transformation, then $S^* = S$, so by induction $S$ has an orthonormal/orthogonal basis of eigenvectors $v_2, \ldots, v_k$. If $v = v_1$, then $\{v_1, \ldots, v_k\}$ yields an orthogonal basis of eigenvectors for $T$. ■

We shall discuss some fundamentally important mathematical consequences of this theorem (over $\mathbb{R}$) at the end of this chapter (probably in lieu of Section 7D in Axler).