Solutions to Exercises 6

8A. Inner products

<u>X1.</u> This amounts to finding solutions to the linear homogeneous system

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is equivalent to the system with matrix $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \ \end{pmatrix} = 0$, which has a 1-dim solution space spanned by $(0,-1,1,0)$.

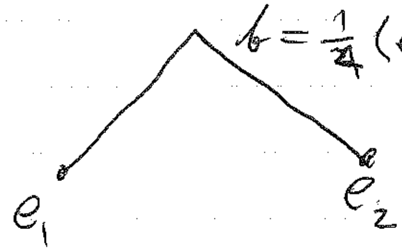
<u>X2.</u> Need to solve $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

This system is equivalent to the one associated to $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. For the latter, we can let x_3 & x_4 vary arbitrarily, and then x_1 & x_2 are forced.

Basis: $(1, -2, 1, 0)$
$(2, -3, 0, 1)$

X3.

$$b = \frac{1}{4}(e_1 + e_2 + e_3 + e_4)$$



$$e_1 - b = \frac{3}{4}e_1 - \frac{1}{4}(e_2 + e_3 + e_4)$$

$$e_2 - b = \frac{3}{4}e_2 - \frac{1}{4}(e_1 + e_3 + e_4)$$

$$\cos\theta = \frac{\langle e_1 - b, e_2 - b\rangle}{|e_1 - b|\,|e_2 - b|}$$

$$|e_j - b| = \sqrt{\frac{9}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$= \frac{(e_1 - b)(e_2 - b)}{\frac{3}{4}} = \frac{-\frac{1}{4}}{\frac{3}{4}} = -\frac{1}{3} \quad \text{so}$$

$\theta \doteq 109.47122\ldots$ degrees.

6.B. Orthogonality and dimension

X1 (a) Let $b = (2,2,3)$ $\quad b = b_1 + b_2$ where

$a = (1,1,0)$

$$b_1 = \frac{b \cdot a}{a \cdot a} a = \frac{4}{2}(1,1,0) = (2,2,0)$$

Hence $b_2 = (0,0,3)$.

(b) Here $a = (1,1,1)$, $b = (1,-1,1)$. In this case

$$b_1 = \frac{b \cdot a}{a \cdot a} a = \frac{1}{3} a = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) \text{ and}$$

$$b_2 = \left(+\tfrac{2}{3}, -\tfrac{4}{3}, +\tfrac{2}{3}\right).$$

X2 (a) Call the three vectors v_1, v_2, v_3.

Then $w_1 = v_1 = (0,0,1,1)$,

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = (0,1,1,0) - \frac{1}{2}(0,0,1,1)$$

$$= \left(0, 1, \tfrac{1}{2}, -\tfrac{1}{2}\right),$$

$$\cancel{w_3 = v_3 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 =}$$

Since $v_3 = v_1$, the vectors w_1 & w_2 span

(b) In this case

$$W_1 = V_1 = (1, 1, 1, 1),$$

$$W_2 = V_2 - \frac{\langle V_2, W_1 \rangle}{\langle W_1, W_1 \rangle} = (-1, 4, 4, 1) - \cancel{\tfrac{8}{4}}^{2}(1,1,1,1)$$

$$= (-3, 2, 2, -1),$$

$$W_3 = V_3 - \frac{\langle V_3, W_1 \rangle}{\langle W_1, W_1 \rangle} W_1 - \frac{\langle V_3, W_2 \rangle}{\langle W_2, W_2 \rangle} W_2 =$$

$$(4, -2, -2, 0) - \cancel{\tfrac{0}{4}}^{0}(1,1,1,1) - \cancel{\tfrac{-20}{18}}_{-10/9}(-3, 2, 2, -1)$$

$$= \left(4 - \frac{10}{3}, -2 + \frac{20}{9}, -2 + \frac{20}{9}, -\frac{10}{9}\right) =$$

$$\left(\frac{2}{3}, \frac{2}{9}, \frac{2}{9}, -\frac{10}{9}\right).$$ Note that

$$9w_3 = (6, 2, 2, -10).$$

When one does computations, it is always a good idea to check the results; i.e., check that

$$\left\{\begin{matrix} \langle w_1, w_2 \rangle \\ \langle w_1, w_3 \rangle \\ \langle w_2, w_3 \rangle \end{matrix}\right\}$$ are all zero.