6C. Orthogonal complements, etc.

X1 Prove that $U^\perp \cap W^\perp = (U+W)^\perp$. (correcting misprint)

① Show $U^\perp \cap W^\perp \subseteq (U+W)^\perp$. Let $x \in U^\perp \cap W^\perp$. Then $\langle x, u\rangle = 0 = \langle x, w\rangle$ for all $u \in U$ & $w \in W$. If $y \in U+W$, write $y = u+w$ with $u \in U$, $w \in W$. Then $\langle x, y\rangle = \langle x, u+w\rangle = \langle x, u\rangle + \langle x, w\rangle = 0+0 = 0$. Hence $x \in (U+W)^\perp$.

② Reverse inclusion: Since $U, W \subseteq U+W$ we have $(U+W)^\perp \subseteq U^\perp, W^\perp$, which means $(U+W)^\perp \subseteq U^\perp \cap W^\perp$.

X2. We know that $E^2 = E$. Therefore if $T = (I-2E)$, then $T^2 = (I-2E)^2 = I - 2(2E) + 2E^2 = I - 4E + 4E = I$.

Also $x \in W \Rightarrow Ex = x \Rightarrow Tx = (I-2E)x = x - 2Ex = x - 2x = -x$. Hence x is an eigenvector for T with eigenvalue -1.

X3. First find an orthogonal basis for W if $v_1 + v_2$ are given as in the problem.

Then $w_1 = v_1 = (1, 1, 0)$ and

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = (0,1,1) - \frac{1}{2}(1,1,0)$$

$$= \left(\tfrac{1}{2}, -\tfrac{1}{2}, 1\right).$$

So an orthogonal basis for W is given by $(1, 1, 0) = x_1$ and $(1, -1, 2) = x_2$.

It follows that

$$E e_j = e_j - \frac{\langle e_j, x_1\rangle}{\langle x_1, x_1\rangle} x_1 - \frac{\langle e_j, x_2\rangle}{\langle x_2, x_2\rangle} x_2$$

Now substitute e_1, e_2, e_3 for e_j.

$$E e_1 = (1,0,0) - \frac{1}{2}(1,1,0) - \frac{1}{6}(1,-1,2) =$$

$$\left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$

$$E e_2 = (0,1,0) - \frac{1}{2}(1,1,0) - \frac{(-1)}{6}(1,-1,2) =$$

$$\left(-\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

Finally,

$$E e_3 = (0,0,1) - \frac{0}{2}(1,1,0) - \frac{2}{6}(1,-1,2) = \left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

So the matrix of E is given by

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

One way to partially check this is to compute E^2 and see if it equals E. If not, there is a mistake in the solution. Also, it turns out that $E = {}^tE$ (transpose) must hold.