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Mathematics 132, Winter 2017, Examination 1

Answer Key

1. [*25 points*] Find an eigenvector for the largest real eigenvalue of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

SOLUTION

The eigenvectors are given by the diagonal entries, so 3 is the largest eigenvalue. Therefore, if A is the matrix displayed above, then the eigenvectors for the eigenvalue 3 are the nonzero vectors in the null space of

$$A - 3I = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

In other words, they are the nontrivial solutions to the equations $-2x + y + z = 0$ and $-y + z = 0$. The leading entries are in the first and second column, so there is a 1–dimensional solution space generated by a vector with $z = 1$ and the other entries dictated by the equations: This means that $y = 1$ and $x = 1$; to summarize, the eigenvectors are the nonzero multiples of $(1, 1, 1)$.■

2. [*25 points*] Suppose that V is a vector space over the real or complex numbers and $S, T : V \to V$ are linear transformations such that $ST = TS$. If c is an eigenvalue of T and W is the subspace consisting of zero and the eigenvectors for c, prove that S maps W into itself (in other words, W is invariant under S).

SOLUTION

We need to show that if $T(v) = cv$ then $T(Sv) = cSv$. This follows because $TSv = STv = Scv = cSv$.■

3. [*25 points*] Let $W \subset \mathbb{R}^3$ be the subspace spanned by $(1,1,0)$ and $(-1,1,2)$. Express the vector $(1,0,1)$ as a sum $\mathbf{x}+\mathbf{y}$, where $\mathbf{x} \in W$ and $\mathbf{y}$ is perpendicular (= orthogonal) to W

SOLUTION

Let $\mathbf{v} = (1,0,1)$, and let $\mathbf{w}_1$ and $\mathbf{w}_2$ be the vectors $(1,1,0)$ and $(-1,1,2)$ respectively. Note that $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal, so there is no need to use the Gram-Schmidt process to find an orthogonal basis for W (we already have one!).

The vector $\mathbf{x}$ is then given by

$$\frac{\langle \mathbf{v}, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w_1} \ + \ \frac{\langle \mathbf{v}, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \mathbf{w_2} \ .$$

If we substitute the numerical values

$$\langle \mathbf{v}, \mathbf{w}_1 \rangle \ = \ 1, \quad \langle \mathbf{w}_1, \mathbf{w}_1 \rangle \ = \ 2, \quad \langle \mathbf{v}, \mathbf{w}_2 \rangle \ = \ 1, \quad \langle \mathbf{w}_2, \mathbf{w}_2 \rangle \ = \ 6$$

into the preceding formula, we see that

$$\mathbf{x} \ = \ \tfrac{1}{2} \cdot (1,1,0) \ + \ \tfrac{1}{6} \cdot (-1,1,2) \ = \ \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right)$$

and therefore

$$\mathbf{y} \ = \ \mathbf{v} \ - \ \mathbf{x} \ = \ \left(\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{2}{3}\right) \ . \blacksquare$$

4. [*25 points*] (a) Find the complex inner product of the vectors $(1, 1+i)$ and $(1, 1-i)$.

(b) Prove the parallelogram identity for real inner product spaces:

$$|\mathbf{x}+\mathbf{y}|^2 \; + \; |\mathbf{x}-\mathbf{y}|^2 \quad = \quad 2 \cdot \left(|\mathbf{x}|^2 \; + \; |\mathbf{y}|^2\right)$$

SOLUTION

(a) The inner product is equal to

$$1 \cdot 1 \; + \; (1+i) \cdot \overline{(1-i)} \quad = \quad 1 \; + \; (1+i)^2 \quad = \quad 2 \; + \; 2i \; - \; 1 \quad = \quad 1 \; + \; 2i \; .$$

(b) We can use the "binomial formula" to express the left hand side as

$$\left(|\mathbf{x}|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + |\mathbf{y}|^2\right) \; + \; \left(|\mathbf{x}|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle + |\mathbf{y}|^2\right)$$

which simplifies to $2 \cdot \left(|\mathbf{x}|^2 + |\mathbf{y}|^2\right)$, which is the right hand side.■