Mathematics 132, Spring 2018, Examination 2

Answer Key

1. [25 *points*] Let V be a finite dimensional inner product space, let W be a (vector) subspace of V, and let $E : V \to V$ be the linear transformation which takes a vector $v \in V$ to its perpendicular projection onto W. Prove that E is self adjoint. [*Hint:* Recall that $v = v_1 + v_2$ where $v_1 \in W$ and $v_2 \in W^\perp$, and this decomposition is unique.]

SOLUTION

The linear transformation E sends $v = v_1 + v_2$ to v_1. Let $x, y \in V$ and write $x = x_1 + x_2$, $y = y_1 + y_2$. We then have

$$\langle Ex, y\rangle \quad = \quad \langle x_1, y_1 + y_2\rangle \quad = \quad \langle x_1, y_1\rangle$$

because x_1 and y_2 are orthogonal. Likewise, we have

$$\langle x, Ey\rangle \quad = \quad \langle x_1 + x_2, y_1\rangle \quad = \quad \langle x_1, y_1\rangle$$

because x_2 and y_1 are orthogonal. Combining these, we have

$$\langle Ex, y\rangle \quad = \quad \langle x, Ey\rangle$$

which implies that E is self adjoint.■

2. [25 *points*] Let V be a finite dimensional inner product space, let c be a scalar, and let $T : V \to V$ be a normal linear transformation. Prove that cT is also normal.

SOLUTION

We need to show that $(cT)(cT)^* = (cT)^*(cT)$. But the standard identities for adjoints imply that $(cT)^* = \overline{c}T^*$ and hence

$$(cT)(cT)^* \quad = \quad c\overline{c}TT^* \quad = \quad \overline{c}cT^*T \quad = \quad (cT)^*(cT)$$

where the second equation is valid because $TT^* = T^*T$ (since T is normal) and $c\overline{c} = \overline{c}c$ (since multiplication of complex numbers is commutative). Therefore cT and its adjoint commute, so by definition this linear transformation is normal.■

3. [*25 points*] Find all Jordan forms (up to rearrangement of blocks) for 5×5 matrices with characteristic polynomial $(1-t)^3(1+t)^2$.

SOLUTION

The eigenvalues of the matrices must be $+1$ and -1, and the sizes of the elementary Jordan blocks for these two eigenvalues must add up to 3 and 2 respectively. Since the possibilities for the blocks associated to the two eigenvalues are independent of each other, we shall consider them separately.

Consider first the eigenvalue $+1$. We shall use the Greedy Algorithm to describe the possibilities systematically.

> The largest possible block is 3×3. If such a block exists in the Jordan form, then there is only one and there is nothing smaller.
>
> If the largest block in the Jordan form is 2×2, the the only option for the remaining blocks is a single 1×1 block.
>
> If the largest block in the Jordan form is 1×1, then all blocks must have this size and there must be three of them.

Hence there are 3 possibilities for the Jordan blocks with eigenvalue $+1$.

Now consider the eigenvalue -1.

> The largest possible block is 2×2. If such a block exists in the Jordan form, then there is only one and there is nothing smaller.
>
> If the largest block in the Jordan form is 1×1, then all blocks must have this size and there must be two of them.

Hence there are 2 possibilities for the Jordan blocks with eigenvalue $+1$.

Combining these, we see that the total number of possible forms is

$$3 \;(= \text{ no. choices for } +1) \;\times\; 2 \;(= \text{ no. choices for } -1) \;=\; 6 \;. \blacksquare$$

4. [25 *points*] Compute the determinant of the following 4×4 matrix:

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

SOLUTION

One way to do this is to start with row operations, using the first row to eliminate all the remaining entries in the first column. This does not affect the determinant, and it yields the following matrix:

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 1.5 & -0.5 & -0.5 \\ 0 & -0.5 & 1.5 & -0.5 \\ 0 & -0.5 & -0.5 & 1.5 \end{pmatrix}$$

One can now use the rule for determinants of matrices in block forms to conclude that the determinant is equal to

$$2 \times \det \begin{pmatrix} 1.5 & -0.5 & -0.5 \\ -0.5 & 1.5 & -0.5 \\ -0.5 & -0.5 & 1.5 \end{pmatrix} = 2 \cdot \frac{1}{8} \times \det \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

We have cleared out the fractions from the 3×3 matrix because it is easier to calculate using integers rather than rational numbers. There are two options now: Either continue with row operations and row reduction or else compute the 3×3 determinant using the standard formula. If we choose the latter we see that the 3×3 determinant is equal to

$$27 + (-1) + (-1) - 3 - 3 - 3 = 16$$

so that $\det A = (2 \cdot 16)/8 = 4$.

Another way to do this is using expansion by minors, say along the last row. This yields the formula

$$\det A = 1 \cdot (-1) \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} + 2 \cdot (+1) \cdot \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = (-1) \cdot 4 + 2 \cdot (8-2-2) = -4+8 = 4$$

which is the same answer obtained by the other method. The example in this problem is an especially good choice for expansion by minors since it has so many zero entries.∎