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Mathematics 132, Winter 2017, Examination 2

Answer Key

1. [*30 points*] (a) Let $W \subset \mathbb{R}^3$ be the span of the vector $(1,1,1)$, and let $E : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which is perpendicular projection onto W. Compute the matrix of E with respect to the three standard unit vectors.

(b) More generally, if V is a finite dimensional inner product space and $E : V \to V$ denotes orthogonal projection onto W, show that $I - E$ is the orthogonal projection onto $W^{\perp}$.

SOLUTION

(a) Let $u_1 = (1,0,0)$, $u_2 = (0,1,0)$, $u_3 = (0,0,1)$ be the standard unit vectors, and let $w = (1,1,1)$. Then the perpendicular projection of u_i onto W is given by

$$\frac{\langle u_i, w\rangle}{\langle w, w\rangle} \cdot w \quad = \quad \frac{1}{3} \cdot (1,1,1)$$

and therefore each column of the matrix for E is given by

$$\frac{1}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

which means that all nine entries of the matrix for E are equal to $\frac{1}{3}$.■

(b) Let $x \in V$ and write $x = y + z$ where $y \in W$ and $z \in W^{\perp}$. Then $Ex = y$ and $(I-E)x = x - Ex = x - y = (y+z) - y = z$; the latter is just the perpendicular projection of x onto $z \in W^{\perp}$, and therefore $I - E$ sends each vector to its perpendicular projection onto that subspace.■

2. [*25 points*] Suppose that V is a vector space over the complex numbers and $T : V \to V$ is an invertible normal linear transformation. Prove that T^{-1} is also normal. [*Hint:* What is the adjoint of T^{-1}?]

SOLUTION

Since T is normal, there is an orthonormal basis Let $u_1, \cdots, u_n$ of eigenvectors for T, so that $Tu_j = c_j u_j$, where of course c_j is the eigenvalue associated to u_j. Since T is invertible, each c_j is nonzero, and therefore T^{-1} is given by $T^{-1}u_j = c_j^{-1}u_j$ for all j. In particular, T^{-1} also has an orthonormal basis of eigenvectors, and therefore it follows that T^{-1} is normal.■

Alternate solution. To answer the question in the hint, we know that $(T^{-1})^* = (T^*)^{-1}$, for if $S = T^{-1}$ we have $S^*T^* = (TS)^* = I^* = I$ and $T^*S^* = (ST)^* = I^* = I$. Therefore we have the string of equations

$$T^{-1}(T^{-1})^* \quad = \quad T^{-1}(T^*)^{-1} \quad = \quad (T^*T)^{-1} \quad = \quad (TT^*)^{-1}$$

because T is normal, and similarly the right hand side is $(T^*)^{-1}T^{-1} \quad = \quad (T^{-1})^*T^{-1}$. Combining these strings of equations, we see that T^{-1} is normal.■

3. [*25 points*] The real symmetric matrix

$$\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$$

has two real eigenvalues, one of which is zero, and an eigenvector for 0 is given by $(1, 1)$. Find the other eigenvalue for this matrix, and find an eigenvector for this eigenvalue.

SOLUTION

The eigenvalues are the roots of $t^2 - 6t = 0$, which are 0 and 6. Furthermore, the eigenvectors for the eigenvalue 6 are the nonzero vectors in the null space of

$$\begin{pmatrix} 3-6 & -3 \\ -3 & 3-6 \end{pmatrix} = \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}$$

and the null space of latter consists of all vectors of the form $(t, -t)$. Any such vector with $t \neq 0$ is an eigenvector for 6. For example, $(1, -1)$ is an eigenvector for this eigenvalue.

Alternatively, one can find an eigenvector for 6 by observing it must be perpendicular to an eigenvector for 0.■

4. [*20 points*] Determine which (if any) of the real symmetric matrices

$$\begin{pmatrix} 2 & 3 \\ 3 & k \end{pmatrix}$$

are positive definite for $k = 3, 4, 5, 6, 7$. Are any of these matrices positive semidefinite but not positive definite? Give reasons for your answer.

SOLUTION

The upper left entry is positive, so the matrix is positive definite if the determinant is positive. On the other hand, if the determinant is negative then the matrix is not positive definite or even positive semidefinite (one eigenvalue is negative, and the other is positive). The determinant of the matrix with k in the lower right corner is $2k - 9$, and this number is positive for $k = 5, 6, 7$ but negative for $k = 3, 4$. Therefore the matrix is positive definite for $k = 5, 6, 7$ but not even positive semidefinite for $k = 3, 4$.■

Extra page for use if needed