MORE EXERCISES FOR CHAPTER 5

Definition. Let V be an vector space over some scalars which will not be specified, and let $T : V \to V$ be a linear transformation. A vector subspace $W \subset V$ is said to be *T-invariant* if $T[W] \subset W$; in other words, if $w \in W$ then $T(w) \in W$.

1. Let $T : V \to V$ be a linear transformation, and let W_1 and W_2 be T-invariant subspaces of V. Prove that $W_1 + W_2$ is also a T-invariant subspace.

2. Suppose that $\dim V = n$ and $S : V \to V$ and $T : V \to V$ are diagonalizable linear transformations (bases of eigenvectors) so that $ST = TS$. Prove that there is a basis of V whose elements are eigenvectors for both S and T. [*Hint:* If v is an eigenvector for T with eigenvalue c, why is the same true for $S(v)$?]

3. Find a basis of eigenvectors for the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

4. Let $\dim V = n$, assume that the scalars are the complex numbers, and let $T : V \to V$ be a linear transformation. Prove that there is a basis (more precisely, an ordered basis) $y_1, \cdots, y_n$ such that for each j we have

$$T(y_j) \quad == \quad \sum_i b_{i,j}\, y_i$$

where the matrix $(b_{i,j})$ is **lower** triangular ($b_{i,j} = 0$ if $i < j$). [*Hint:* One approach is to extract this result from the result stating that there is an ordered basis $x_1, \cdots, x_n$ such that the matrix of T with respect to the latter is upper triangular. We can then choose the new basis to be a reordering of $x_1, \cdots, x_n$.]