MORE EXERCISES FOR CHAPTER 6

1. Let V be the space of real 2×1 matrices with the analog of the usual inner product on $\mathbb{R}^n$ (sum of the products of the corresponding coordinates), and define $\varphi : V \times V \to \mathbb{R}$ by the formula

$$\varphi(X, Y) \quad = \quad X^* A Y$$

where X^* denotes the transpose of X and A is the following 2×2 matrix:

$$\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

Prove that φ defines an inner product. [*Hints:* Show that A has an orthogonal basis of eigenvectors $\{w_1, w_2\}$ with eigenvalues 8 and 2. Find explicit eigenvectors. Express X as a linear combination $\xi_1 w_1 + \xi_2 w_2$ for suitable scalars ξ_j and verify that $X^* A X \;=\; 8\,\xi_1^2 \;+\; 2\,\xi_j^2$; the scalars ξ_j should be given explicitly in terms of the entries for X. The other properties of an inner product follow from corresponding identities for matrix multiplication and the fact that A is its own transpose.]

2. Suppose that φ_0 and φ_1 are inner products on some real or complex vector space V, and assume that $0 < t < 1$. Prove that for each such t the function

$$\varphi_t(x, y) \quad = \quad t\varphi_0(x, y) \;+\; (1 - t)\varphi_1(x, y)$$

defines an inner product on V.

3. Let V and W be real or complex inner product spaces with orthonormal bases $\{a_1, \;\cdots\;, a_n\}$ and $\{b_1, \;\cdots\;, b_m\}$ respectively, and let $T : V \to W$ be a linear transformation whose effect on basis vectors is given by $T(a_j) = \sum_i \; p_{i,j} b_i$. Prove that $p_{i,j} = \langle T(a_j), b_i \rangle$ for all i and j.

4. Find the distance from the vector $(2, 3, 4, 5) \in \mathbb{R}^4$ to the hyperplane (3-dimensional subspace through the origin) with defining equation $x_1 + x_2 + x_3 + x_4 = 0$. [*Hints:* First find a basis for the subspace using row operations on matrices, then find an orthonormal basis by the Gram-Schmidt process.]

5. Let $W \subset \mathbb{C}^3$ be the subspace with basis $(1, 0, i)$ and $(1, 2, 1)$. Find a basis for $W^\perp$.

6. Let V be an inner product space over the real or complex numbers, let W be a finite-dimensional subspace of V, and let x be a vector in V such that $x \notin W$. Prove that there is some $y \in W^\perp$ such that $\langle x, y \rangle \neq 0$.