MORE EXERCISES FOR SECTION 7A

1. Let V and W be inner product spaces over the real or complex numbers, and suppose that $T : V \to W$ be a linear transformation with adjoint T^*. Prove that T^*T and TT^* are positive semidefinite, and show that the ranks of both are equal to the rank of T.

2. Let S and T be self adjoint linear transformations on the inner product space V. Prove that ST is self adjoint if and only if $ST = TS$.

EXAMPLE. Let f be the real valued function on the real line defined by $f(x) = \exp(-1/x^2)$ if $x > 0$ and $f(0) = 0$ if $x \leq 0$. Then clearly f is infinitely differentiable for $x \neq 0$, and repeated applications of L'Hospital's Rule imply that the n^{th} derivative $f^{(n)}(0)$ is 0 for all n (you need not verify this). If we then define $b(x) = f(x) \cdot f(1-x)$ (*i.e.*, a "bump function") it follows that b is infinitely differentiable with $b(x) = 0$ if $x \notin [0,1]$, but $b(x) > 0$ on the open interval $(0,1)$. Using changes of variables and linear combinations, one can use this example to construct many other functions in V.

3. Let V be the vector space of all real valued functions on $[0,1]$ which are infinitely differentiable and satisfy $f^{(n)}(0) = f^{(n)}(1) = 0$, and define an inner product by the formula

$$\langle\, f,\, g \,\rangle \quad = \quad \int_0^1 \ f(t)g(t)\, dt \ .$$

The previous discussion implies that V is a nontrivial vector space (in fact, it is infinite dimensional). Explain why differentiation induces a linear transformation D from V to itself, and find the adjoint of D. [*Hint:* Integration by parts.]