STILL MORE EXERCISES FOR CHAPTER 7

1. Let V be a real inner product space, let $u \in V$ be a unit vector ($|u| = 1$), and let $S : V \to V$ be the linear transformation

$$S(v) \quad = \quad v \ - \ 2\langle \, v, \, u \, \rangle \, u \ .$$

Prove that S is orthogonal and in fact $S^2 = I$. Furthermore, show that the eigenspace of -1 is spanned by u.

2. Give an example of a linear transformation (or square matrix) A such that A^2 is normal but A is not normal.

3. Prove that a self adjoint linear transformation T on a finite dimensional real inner product space V is positive semidefinite if and only if $A = B^*B$ for some $B : V \to V$.

4. Suppose that A is a symmetric square matrix over the reals. Prove that there is a real symmetric matrix B so that $B^3 = A$. [*Hint:* Look at the induced self adjoint linear transformation.]

5. Let T and S be self adjoint positive definite linear transformations on the inner product space V. Prove the following:

(a) $S + T$ is positive definite.

(b) If c is a positive real number then cT is positive definite.

(c) T^{-1} is positive definite.

(d) If $U : V \to V$ is self adjoint then there is some real number $a > 0$ such that $U + aI$ is positive definite.

6. Let V be a finite dimensional complex inner product space, and let $T : V \to V$ be self adjoint. Prove the following:

(a) $|v + iTv| = |v - iTv|$ for all $v \in V$.

(b) The linear transformations $I \pm iT$ are invertible.

(c) The **Cayley transform**

$$U \quad = \quad (I \ - \ i\,T)^\circ(I \ + \ i\,T)^{-1}$$

is unitary.