EXERCISES FOR CHAPTER 8

8A. Primary decomposition

Exercises in Axler: 1, 5, 8, 9, 13

Additional exercises:

X1. Let A be a strictly upper triangular 3×3 matrix of the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

where $abc \neq 0$ (hence all three entries are nonzero). General considerations imply that $A^3 = 0$. Is it possible that $A^2 = 0$? Either prove that it is never zero or else give a counterexample.

X2. Find the nonzero subspaces V_λ for the following matrix, and find the eigenvectors associated to each of the eigenvalues:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

8B. Rational decomposition of nilpotent transformations

Exercises in Axler: 1, 3, 6, 9, 11

Additional exercises:

X1. An $n \times n$ matrix A is said to be a *Heisenberg matrix* if it is upper triangular, its diagonal entries are all equal to 1, and the only nonzero entries off the diagonal appear in the first row and last column.

(a) Translate the preceding description of such matrices into algebraic conditions on the entries $a_{i,j}$ of the matrix A.

(b) Show that 1 is the only eigenvalue of a Heisenberg matrix, and give examples of 5×5 Heisenberg matrices where the space of eigenvectors is k-dimensional, where $k = n - 1$ or $n - 2$.

(c) Explain why $(A - I)^3 = 0$ if A is a Heisenberg matrix.

X2. Suppose that N is a stricly upper triangular $n \times n$ matrix. Prove that $I + N$ is invertible and its inverse is given by $I + N + N^2 + \cdots + N^{n-1}$.

8C–8D. Jordan Canonical Form

Exercises in Axler, Section 8C: 1, 5, 6, 7, 11

Exercises in Axler, Section $8D$*:* 4, 5 assuming that $V = V_\lambda$ for the eigenvalue λ and that the matrix for $T : V \to V$ is an elementary Jordan matrix.

Additional exercises:

X1. Let V be a finite dimensional complex vector space, let $T : V \to V$ be a (complex-) linear transformation, and assume that $V = V_\lambda$ for some λ. Determine the number of inequivalent Jordan form possibilities for T if $\dim V = 5$.

X2. Determine the number of inequivalent possibilities for the Jordan form of a matrix whose eigenvalues are 2 and 3.

X3. Suppose that A is an $n \times n$ unitriangular matrix (upper triangular, with all diagonal entries equal to 1), and write $A = I + N$ where N is strictly upper triangular. Compute $A^4 = (I + N)^4$ if A is an elementary 5×5 Jordan matrix.

Warning

If A and B are $n \times n$ matrices, then the usual Binomial Formula for $(A + B)^k$ can only be used when A and B commute (*i.e.*, $AB = BA$).