MORE EXERCISES FOR CHAPTER 10

1. Let V be a finite dimensional vector space. A linear transformation $T : V \to V$ is said to be an **involution** if T^2 is the identity. Since we are working with real or complex scalars, we know that $1 \neq -1$.

(a) Denote the images of $I \pm T$ by V_+ and V_- respectively. Show that V is a direct sum of V_+ and V_-, and that $Tx = \pm x$ if $x \in V_\pm$. [*Hint:* Consider the vectors $x_\pm = \frac{1}{2}(x \pm Tx)$.]

(b) Express the trace of T in terms of the dimensions of V and V_+.

2. Suppose that A is a 2×2 matrix which does not have a basis of eigenvectors over the real or complex numbers. Find a nontrivial second degree equation relating the trace and determinant of A.

3. Prove that the determinant of the 3×3 *Vandermonde matrix*, which is named after Alexandre-Théophile Vandermonde (1735–1796):

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$$

is equal to $(b-a)(c-a)(c-b)$.

NOTE. See `http://math.uga.edu/ ~pete/Skoufranis12.pdf` for generalizations of the Vandermonde matrix and determinant formula to higher dimensions.

4. Recall the standard formula for the cross product of two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$:

$$a \times b = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

(a) Verify the "BAC—CAB" rule for threefold cross products:

$$A \times (B \times C) = B\,(A \cdot C) - C\,(A \cdot B)$$

Here the scalar multiplication is written on the right as aid to memorizing the formula.

(b) Verify that the cross product satisfies the *Jacobi identity*:

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$$