

5A - Eigen{values} {vectors} and invariant subspaces

The main goal of Ch. 5 is to understand linear transformations better. In particular, we want to show that one can often change bases for a finite dimensional vector space V so that a linear transformation $T: V \rightarrow V$ looks simpler in the new basis. The focal point will be linear transformations for which one can find a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ ($n = \dim V$) such that $Tv_i = cv_i$ for some scalars c_i .

We shall say that T is diagonalizable if such a basis exists, for if we define the $n \times n$ matrix of some $S: V \rightarrow V$ by

$$Sv_i = \sum_j a_{ij} v_j,$$

then the matrix of $S = T$ as above will only have nonzero entries down the main diagonal:

$$\begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}.$$

Not every $T: V \rightarrow V$ is diagonalizable, but "most" turn out to be so in a sense we shall describe later.

Here are some formalities.

Def. Let $T: V \rightarrow V$ (possibly $\dim V = \infty$).

A nonzero $v \in V$ is said to be an eigenvector for T if $Tv = cv$ for some scalar c .

The number c is called an eigenvalue for T .

Some writers who prefer not to use an amalgamation of German and English syllables write characteristic $\left\{ \begin{matrix} \text{value} \\ \text{vector} \end{matrix} \right\}$ or proper $\{ - \}$ instead.

If c is an eigenvalue, then the eigenspace $V_{c,T}$ or V_c is $\{0\}$ together with all eigenvectors for c (with respect to T , of course).

General facts (i) V_c is a (vector) subspace.
(ii) T maps V_c into itself (i.e., V_c is a T -invariant subspace).

VERIFICATIONS. First, $T0 = 0 = c0$,
 so $V_c = \text{all } v \in V \text{ such that } Tv = cv.$

(i) $Tv = cv + Tw = cw \Rightarrow T(v+w) =$
 $Tv + Tw = cv + cw = c(v+w)$, and
 $Tv = cv, a \in F \Rightarrow T(av) = aTv =$
 $a(cv) = c(av). \blacksquare$ (V_c)

(ii) Suppose $v \in V_c$, so that $Tv = cv.$

~~Then $T(cv) = c(Tv) = c(cv) = c^2v$
 $= c(cv),$ so~~

Then $Tv = cv$ also lies in V_c because
 $T(Tv) = T(cv) = cTv. \blacksquare$

Proposition Let $T: V \rightarrow V$. Then c is
 an eigenvalue for $T \iff \text{Kernel}(T - cI)$ is
 nonzero.

Proof. $(\iff) Tv = cv \text{ with } v \neq 0 \iff (T - cI)v = 0$
 $\iff 0 \neq v \in \text{Kernel } T - cI. \blacksquare$

Cor. If V is fin dim, c is an eigenvalue \iff
 $(T - cI)$ is not 1-1, and also $\iff (T - cI)$ is not onto.

Finding eigenvalues + eigenvectors.

Example in Axler proves rigorously that T_A has no real eigenvalues if

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ but it has } \text{complex} \text{ eigenvalues.}$$

Here are examples of matrices with bases of real eigenvectors.

1. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. To find eigenvalues we must find scalars c such that

$$\begin{pmatrix} 1-c & 2 \\ 2 & 1-c \end{pmatrix} \text{ is not invertible. Do this}$$

by trying to put the matrix into row-reduced echelon form. Alternatively, in the 2×2 case

it suffices to show that one row is a scalar multiple of the other: $(1-c, 2) = k(2, 1-c)$

or $\frac{1-c}{2} = k = \frac{2}{1-c}$. Solving this proportionality

equation, we get $(1-c)(1-c) = 4$, or

$c^2 - 2c - 3 = 0$. This factors as $(c+1)(c-3)$
 so eigenvalues exist and are 3, -1.

We can now check that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector
 for +3, and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for -1.

2. $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$, so $A - cI = \begin{pmatrix} -c & -2 \\ 1 & 3-c \end{pmatrix}$.

In this case the proportionality condition
 becomes $c^2 - 3c = -2$ or $c^2 - 3c + 2 = 0$

or $(c-2)(c-1) = 0$. The eigenvectors for
 +2 are the non-zero solutions for $\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

[non-zero multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$] and the eigenvectors
 for +1 are the non-zero ~~multiples~~ ^{solutions} of

$$\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad [\text{multiples of } \begin{pmatrix} 2 \\ -1 \end{pmatrix}].$$

General rule for 2x2 case. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The eigenvalues are those r such that
 $(a-r, b)$ and $(c, d-r)$ are lin dep.,
 or equivalently the determinant

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} \text{ is zero.}$$

(We assume 2×2 and 3×3 determinants
 are known from "College Algebra")

Final remarks

Theorem. If v_1, \dots, v_k are eigenvectors
 for distinct eigenvalues c_1, \dots, c_k then
 $\{v_1, \dots, v_k\}$ is linearly independent.

Proof $k=1$, no problem. Assume true
 for $(k-1)$ vectors & proceed by induction.
 If the set is not linearly independent, then,
 say, v_k is a linear combination of the
 previous vectors, so that

$$v_k = \sum_{j < k} a_j v_j. \text{ Then}$$

$$c_k v_k = \sum c_{kj} a_j v_j \text{ on one hand, but also}$$

$$c_k v_k = T v_k = \sum a_j T v_j = \sum c_j a_j v_j.$$

Since $v_k \neq 0$, some $a_{j_0} \neq 0$. By linear independence, $c_k a_{j_0} = c_{j_0} a_{j_0}$, so that $c_k = c_{j_0}$, which contradicts the assumption that c_1, \dots, c_k were distinct. Hence $\{v_1, \dots, v_k\}$ is linearly independent, completing the inductive step. ■

Corollary If $\dim V = n$, then T has at most n distinct eigenvalues. ■

If there are in fact n distinct eigenvalues, then T must be diagonalizable. ■

A non diagonalizable example

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad a = \text{any scalar } a \neq 0$$

Look at $|A - \lambda I| = \text{determinant} \begin{vmatrix} 1-c & a \\ 0 & 1-c \end{vmatrix}$
 $= (1-c)^2$. So $c=1$ is the only eigenvalue.

Then $(A - I) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \Rightarrow \text{Kernel } A - I$
spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So there is no basis of
eigen vectors in this example, either over
 \mathbb{R} or \mathbb{C} .