5C. More on diagonalization

Thm. If $T: V \to V$ has $n = \dim V$ $\boxed{\text{distinct}}$ eigenvalues, then T is diagonalizable.

Proof. Let $v_1, \ldots, v_n$ be eigenvectors for the distinct eigenvalues $c_1, \ldots, c_n$. Then we know $\{v_1, \ldots, v_n\}$ is linearly independent, and since $n = \dim V$ this set must form a basis for V. ∎

Axler, Example 5.45

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$$

By a previous result, the eigenvalues are 2, 5, 8.

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 6 \end{pmatrix} \qquad A - 5I = \begin{pmatrix} -3 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

eigenvector for this value is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

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$$A - 8I = \begin{pmatrix} -6 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{eigenvector} \quad \begin{pmatrix} 1 \\ +6 \\ +6 \end{pmatrix}.$$

<u>Problem.</u> For what real matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can one find a basis of real eigenvectors.

<u>Solution</u> Eigenvalues are the roots of $t^2 - (a+d)t + (ad-bc) = 0$, and this equation has two real roots if the discriminant $B^2 - 4AC$ is positive, so the answer is that there are two real roots $\Leftrightarrow$

$$(a+d)^2 > 4(ad-bc).$$

Over the complex numbers, there are two roots $\Leftrightarrow (a+d)^2 \neq 4(ad-bc)$.

Iterations of linear transformations

There are many situations in which we have a vector $v \in V$, a linear transformation $T : V \to V$, and a reason for considering the iterated powers of T acting on v:

$$v,\ Tv,\ T^2 v, \ldots$$

If we have a basis of eigenvectors for T we can do so as follows.

① Find [the] eigenvectors $x_1, \ldots, x_n$ forming a basis along with their associated eigenvalues c_i.

② Find the coefficients a_i such that $v = \sum_i a_i x_i$

③ Conclude that $T^k v = \sum_i a_i T^k x_i = \sum_i a_i c_i^k x_i$.

Previous example continued

$A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Have seen $A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and

$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Find $A^k\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Step ① is already completed.

For Step 2, write $\begin{pmatrix}1\\0\end{pmatrix} = p\begin{pmatrix}2\\-1\end{pmatrix} + q\begin{pmatrix}1\\-1\end{pmatrix}$ & solve for p, q. Use any valid method & get $p=1$, $q=-1$: $\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix}$.

Then $A^k\begin{pmatrix}1\\0\end{pmatrix} = 2^k\begin{pmatrix}2\\-1\end{pmatrix} - \cancel{1^k}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2^{k+1}-1\\-2^k-1\end{pmatrix}$

Application to Amortized Loans

<u>Setting.</u> A sum S of money is borrowed for Y years at an annual interest rate of q percent. The money is to be repaid in equal monthly installments, with interest charged only on the remaining balance. Find the monthly payments P.

Let $N = 12Y$ be the number of payments, and let $r = \frac{q}{1200}$ (monthly interest converted to fractions). Then we have

$$x_{k+1} = x_k - \underbrace{P}_{\substack{\text{subtract}\\\text{payment}}} + \underbrace{r x_k}_{\substack{\text{add back}\\\text{interest charges}}}$$

where x_k = monthly balance at month k.

We also know that $x_0 = S$ and $x_N = 0$.

Rewrite everything in matrix form:

$$\begin{pmatrix} x_{k+1} \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1+r & -P \\ 0 & 1 \end{pmatrix}}_{\text{"}A\text{"}} \begin{pmatrix} x_k \\ 1 \end{pmatrix}$$

Then we want to solve for P. This requires finding a formula for $\begin{pmatrix} x_k \\ 1 \end{pmatrix} = A^k \begin{pmatrix} x_0 \\ 1 \end{pmatrix}$.

① Find a basis of eigenvectors for A.

The eigenvalues are easy to find because $1+r$ and 1 are distinct ($r>0$, of course).

They eigenvectors are $V_1 = \begin{pmatrix} P \\ r \end{pmatrix}$ $V_{1+r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

[Solve $(A - cI)(W) = 0$ for $c = 1, 1+r$]

② Express $\begin{pmatrix} x_0 \\ 1 \end{pmatrix} = \begin{pmatrix} S \\ 1 \end{pmatrix}$ as a linear combination of these. In fact, we have

$$\begin{pmatrix} x_0 \\ 1 \end{pmatrix} = \frac{1}{r} V_1 + \left[S - \frac{P}{r} \right] V_{1+r}.$$

This immediately yields

$$\begin{pmatrix} x_k \\ 1 \end{pmatrix} = A^k \begin{pmatrix} S \\ 1 \end{pmatrix} = \frac{1}{r}\begin{pmatrix} P \\ r \end{pmatrix} + (1+r)^k \left[S - \frac{P}{r} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

but we still need to find P. To do so, use the condition $x_N = 0$, so that

$$\frac{P}{r} + (1+r)^n \left[S - \frac{P}{r} \right] \cancel{1} = 0.$$

Solving this linear eqn. for P, we get

$$P = \frac{r S (1+r)^N}{(1+r)^N - 1}. \blacksquare$$