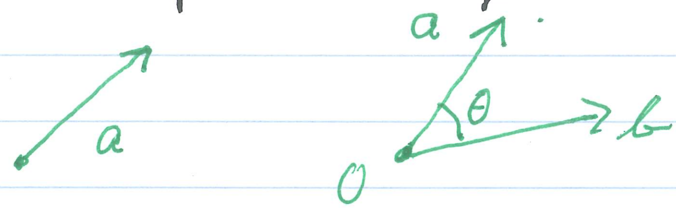
6A. Inner (or Dot) products

If $n = 2$ or 3, the DOT PRODUCT on $\mathbb{R}^n$ is given as follows. Given a, b in $\mathbb{R}^n$, write their coordinates as a_i + b_i respectively. Then

$$a \cdot b = \langle a, b \rangle = \sum_{i=1}^{2 \text{ or } 3} a_i b_i .$$

MORE COMMON IN ADV. MATH.

We can use dot products to express important geometric information algebraically.



<u>Length</u> $|a| = \sqrt{a \cdot a}$ (Pythagorean Thm.)

positive, zero $\Leftrightarrow a = 0$

<u>Angle</u> $\cos\theta = \dfrac{a \cdot b}{|a| \cdot |b|}$ (Law of Cosines)

Dot products satisfy simple identities

$$a \cdot b = b \cdot a, \quad k(a \cdot b) = (ka) \cdot b = a \cdot (kb)$$

$$(a_1 + a_2) \cdot (b_1 + b_2) = a_1 \cdot b_1 + a_2 \cdot b_1 + a_1 \cdot b_2 + a_2 \cdot b_2 .$$

$a \cdot a \geq 0$, with equality $\Leftrightarrow a = 0$.

Other identities like $0 \cdot a = 0 = a \cdot 0$ follow from these.

In $\mathbb{R}^3$ the cross product is useful, but it doesn't extend to higher dimensions in a simple fashion.

Inner product in $\mathbb{R}^n$ If $a, b \in \mathbb{R}^n$, then $\langle a, b \rangle = \sum_{i=1}^{n} a_i b_i$.

It is an "exercise in bookkeeping" to check that $\langle a, b \rangle$ has all the previously listed alg-properties of the dot product. We shall see it also has the geometric properties of the dot product given above.

Length $|a| = \sqrt{a \cdot a}$ (unique nonneg sqrt).

Angle $\theta = \arccos \dfrac{a \cdot b}{|a| \cdot |b|}$.

Immediate question. How do we know the latter makes sense? — In particular, why is the quotient always between -1 and $+1$?

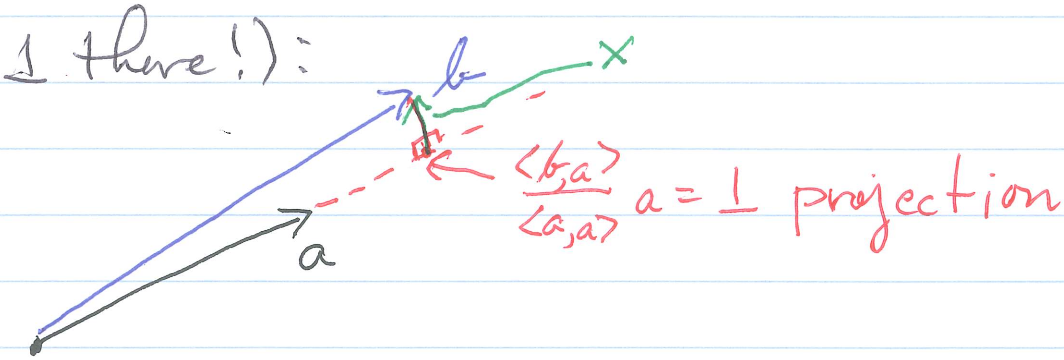
Cauchy-Schwarz-Buniakovski Inequality

$$|a \cdot b| \le |a| \cdot |b| \quad \text{with}$$

equality $\Longleftrightarrow$ a & b are linearly dependent.

The conclusion follows immediately if $a = 0$ or $b = 0$, so assume in the derivation that $a, b \neq 0$.

Derivation. Let $x = b - \frac{\langle b, a\rangle}{\langle a, a\rangle} a$. In $\mathbb{R}^2$ or $\mathbb{R}^3$, the vector x is perpendicular to the line containing 0 and a (also in higher dims, once we define $\perp$ there!):



We know that $\langle x, x\rangle \geq 0$. Now substitute the definition of x to get $\left\langle b - \frac{\langle b,a\rangle}{\langle a,a\rangle} a,\ b - \frac{\langle b,a\rangle}{\langle a,a\rangle} \right\rangle =$
$|b|^2 - 2\frac{\langle b,a\rangle}{\langle a,a\rangle}\langle b,a\rangle + \frac{\langle b,a\rangle^2}{|a|^4}|a|^2 \geq 0.$

We can rewrite this as $|b|^2|a|^2 \geq \langle a, b\rangle^2$, and taking square roots yields the CSB $\leq$. $\blacksquare$

Inner product on $\mathbb{C}^n$ We have to give up something, and experience shows it's best to define a complex inner product so that $\langle v,v\rangle$ is the length of v when it is viewed as a vector in $\mathbb{R}^{2n}$:

$$\text{complex } \langle v,w\rangle = \sum_{j=1}^{n} v_j \overline{w_j}$$ ← conjugation

If $v = x+iy$ then $\langle v,v\rangle$ becomes

$$\sum v_j \overline{v_j} = \sum x_j^2 + y_j^2,$$ as desired.

This leads to a modified commutative law:

$$\langle w,v\rangle = \overline{\langle v,w\rangle}$$

and also a modified homogeneity property:

$$c\langle v,w\rangle = \langle cv,w\rangle = \langle v,\overline{c}w\rangle \text{ or}$$

equivalently, $\langle v, cw\rangle = \overline{c}\langle v,w\rangle$.

~~Norms (lengths)~~

<u>Abstraction</u> An <u>inner product space</u> over $\mathbb{R}$ or $\mathbb{C}$ is a vsp V plus an inner product $\langle\, ,\, \rangle : V \times V \longrightarrow \mathbb{C}$ which satisfies the given properties.

See Axler, p. 166 for examples.

<u>Still more</u>: If $d_i \neq 0$ for $i = 1, \ldots, n$ then $\sum |d_j|^2 v_j \overline{w_j}$ defines an inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$ ($\bar{a} = a$ if a is real).

Note that the length $|v| = \sqrt{\langle v, v\rangle}$ of a vector in an inner product space will satisfy $|v| = 0$, with $|v| = 0 \Leftrightarrow v = 0$
$|cv| = |c| \cdot |v|$.

MORE GEOMETRY.

Two vectors are $\left\{ \begin{matrix} \text{perpendicular} \\ \text{orthogonal} \end{matrix} \right\}$ if $\langle v, w \rangle = 0$.
$(\Leftrightarrow \langle w, v\rangle = 0)$.

In fact, the Cauchy-Schwarz-Buniakovsky inequality extends to all inner product spaces!

As before, need only consider the case $a, b \neq 0$.

Derivation

Write $x = b - \frac{\langle b,a\rangle}{\langle a,a\rangle} a$. Then

$$\langle x,a\rangle = \langle b,a\rangle - \frac{\langle b,a\rangle}{\langle a,a\rangle}\langle a,a\rangle = 0,$$

so $|b|^2 = \left\langle x + \frac{\langle b,a\rangle}{\langle a,a\rangle} a,\ x + \frac{\langle b,a\rangle}{\langle a,a\rangle} a\right\rangle =$

$$|x|^2 + \frac{\langle b,a\rangle}{\langle a,a\rangle}\underset{0}{\langle x,a\rangle} + \frac{\overline{\langle b,a\rangle}}{\langle a,a\rangle}\underset{0}{\langle x,a\rangle} + \frac{|\langle a,b\rangle|^2}{|a|^4}|a|^2$$

Hence $0 \le |x|^2 = |b|^2 - \frac{|\langle a,b\rangle|^2}{|a|^2}$ or

$|b|^2 \ge \frac{|\langle a,b\rangle|^2}{|a|^4}$, which means

$|a|^2|b|^2 \ge |\langle a,b\rangle|^2$. Take sq roots to get the desired inequality.

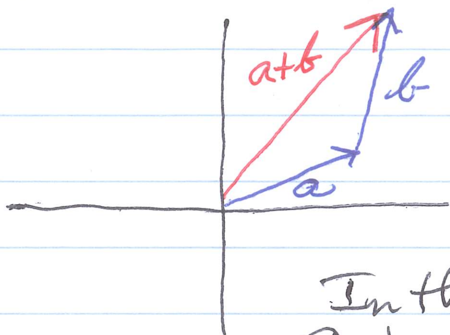
Suppose equality holds. Then $|x|^2$ must be 0, which means b is a scalar multiple of a.

Conversely, $b = k\cdot a \Rightarrow |\langle a,\underset{ka}{b}\rangle|^2 = |k|^2|a|^2 =$

$|a|^2|ka|^2 = |a|^2|b|^2$. ∎

Here is an even more geometrically motivated result:

TRIANGLE INEQUALITY



Over $\mathbb{R}$ or $\mathbb{C}$,

$$|a+b| \leq |a| + |b|.$$

In this picture strict $\leq$ holds.
But $=$ holds if, say, $a = b$.

Derivation Need only prove squares are unequal in the same order.

$$|a+b|^2 = \langle a+b, a+b\rangle = |a|^2 + \langle a,b\rangle + \langle b,a\rangle + |b|^2 =$$

(since $\langle b,a\rangle = \overline{\langle a,b\rangle}$). $|a|^2 + 2\,\text{real}\langle a,b\rangle + |b|^2 =$

(since real part $x+yi \leq |x+yi|$) $|a|^2 + 2|\langle a,b\rangle| + |b|^2 \leq$

(C-S-B inequality) $|a|^2 + 2|a||b| + |b|^2 =$

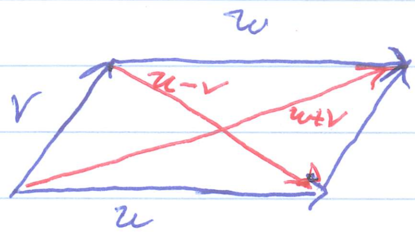
$$(|a| + |b|)^2.$$

To finish, take sq roots of the first & last expressions. ∎

If a & b are linearly independent, then strict inequality holds because $|\langle a, b\rangle| <$ $|a| \cdot |b|$ in such cases.

One more result:

Parallelogram Law



Over $\mathbb{R}$,

$$|u+v|^2 + |u-v|^2 = 2\left(|u|^2 + |v|^2\right)$$

Derivation The left hand side is

$$|u|^2 + \cancel{2\langle u,v\rangle} + |v|^2 + |u|^2 - \cancel{2\langle u,v\rangle} + |v|^2$$

$$= 2|u|^2 + 2|v|^2. \blacksquare$$

See Axler, p. 174, for a proof that works in the complex case.