8A. Primary decomposition

All vector spaces finite dim'l, over $\mathbb{C}$

We want to describe standard (or canonical) forms for describing a linear transformation $T: V \to V$ which does not have a basis of eigenvectors.

Let's begin by assuming $T: V \to V$ is in triangular form for some ordered basis $\{w_1, \dots, w_n\}$: $Tw_j = \sum_{i \le j} a_{ij} w_i$

<u>Lemma 1</u> Let $f(t) = \prod (t - a_{ii})$. Then $f(T)$ is strictly upper triangular; i.e., all diagonal entries are zero.

<u>Proof.</u> Notice first that each T^k is in triangular form. Also, the (i,i) diagonal entry of $f(T)$ is $f(a_{ii})$, which is zero. $\boxed{}$

<u>Lemma 2</u> Suppose that $T : V \to V$ is strictly upper triangular. Then $T^n = 0$, where $n = \dim V$.

<u>Proof.</u> Let $V_k = \text{Span}\{v_1, \dots, v_k\}$, $1 \le k \le n$. Then $T[V_k] \subseteq [V_{k-1}]$ if $k > 0$ and $T[V_1] = \underset{``V_0"}{\{0\}}$. By induction this means that

$$T^j[V = V_n] \subseteq V_{n-j}, \text{ so } T[V_n] \subseteq \{0\}. \ \blacksquare$$

<u>Lemma 3</u> Let $\lambda_1, \dots, \lambda_r$ be the eigenvalues of T. Then there is some $M > 0$ such that $\prod_j (T - \lambda_j I)^M = 0$ on V.

<u>Proof</u> By the first two lemmas we know that if $g(t) = f(t)^n$, then $g(T) = 0$. Now the constants in the linear factors $(t - a_{ii})$ are the eigenvalues of T (this was an exercise). Suppose now that we

form $h(t)$ from $f(t)$ by adding a product of linear factors $(t-\lambda_j)$, say $f_1(t)$, so that $h(t) = f_1(t)f(t) = \prod_j (t-\lambda_j)^{M_0}$. Then

just n

$$h(t)^n = f_1(t)^n f(t)^n \Rightarrow$$

$$\underset{\underset{\prod_j (T-\lambda_j I)^{nM_0}}{\shortparallel}}{h(T)^n} = f_1(T)^n f(T)^n = f_1(T)^n \cdot 0 = 0$$

■

<u>Def.</u> $T: V \to V$ as before. Let V_λ = all $v \in V$ s.t. for some $m \geq 1$, $(T-\lambda I)^m v = 0$.

<u>Lemma 4</u> V_λ is a vector subspace of V, and T maps V_λ into itself.

<u>Proof</u> $(T-\lambda I)^{m(1)} v_1 = 0 = (T-\lambda I)^{m(2)} v_2 = 0$

$\Rightarrow (T-\lambda I)^{m(1)+m(2)} v_1 + v_2 = 0$, $(T-\lambda I)^{m(1)} cv = 0$.

Also, if $(T-\lambda I)^m v = 0$, claim

$(T-\lambda I)^m Tv = 0$. This follows because the left hand side equals $T(T-\lambda I)^m v = T0 = 0$. ∎

PRIMARY DECOMPOSITION THEOREM.

Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of T. Then every $v \in V$ has a unique decomposition as $\sum_j v_j$ where $v_j \in V_{\lambda_j}$.

Manipulations with polynomials have played a significant role thus far in this section. The proof of the theorem will require still more use of polynomials.

__Lemma 5.__ There is some $q > 0$ such that $(T-\lambda I)^q$ sends V_λ to $\{0\}$.

__Proof.__ Let $x_1, \ldots, x_p$ be a basis for V_λ, so for each i, $(T-\lambda I)^{q_i} x_i = 0$ for some q_i. Therefore $q = \max\{q_i\}$ implies $(T-\lambda I)^q x_i = 0$, all i. Since $\{x_i\}$ is a basis, $(T-\lambda I)^q = 0$ on V_λ. ∎

<u>Note</u> We might as well take the exponent M in Lemma 3 to be so large that $(T - \lambda_j I)^M$ is zero on V_{λ_j} for each j.

Digression on polynomials

PRINCIPAL IDEAL PROPERTY. Let J be a nonempty set of polynomials over a field $\mathbb{F}$ such that

$$f, g \in J \Rightarrow f+g \in J,$$

$$f \in J,\ h \in \mathbb{F}[t] \Rightarrow h \cdot f \in J.$$

POLYNOMIALS

Then either $J = \{0\}$ or J is the set of all polynomials which are multiples of a fixed polynomial of least degree.

This is a consequence of the "long division of polynomials" discussed in Chapter 4 of Axler.

CONSEQUENCE. Suppose $p_1, \ldots, p_k \in \mathbb{C}[t]$ such that each p_j is a product of linear factors and no polynomial $(t-c)$ divides every p_j. Then there are polynomials s_j such that

$$\sum s_j p_j = 1.$$

EXAMPLE. Take $(t-a), (t-b)$ where $a \neq b$. Then $1 = \frac{(t-a)-(t-b)}{b-a}$.

PROOF OF THE PRIMARY DECOMPOSITION THEOREM

Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of T, and choose N such that

$$p(T) = \prod_{j=1}^{r} (T - \lambda_j I)^N = 0 \text{ on } V.$$

Let $q_j(t) = \prod_{k \neq j} (t - \lambda_k)^N$. Then the polynomials q_j have no common linear factors, so the Principal Ideal Property implies that $1 = \sum s_j(t) q_j(t)$ for suitable polynomials $s_j(t)$. It follows that

$$v = \sum s_j(T) q_j(T) v. \text{ for all } v \in V.$$

If $v_j = s_j(T) q_j(T) v$, then

$$(T-\lambda_j I)^N v = s_j(T)\, q_j(T)(T-\lambda_j I)^N v$$

$$= s_j(T)\, p(T) v = s_j(T)\, 0 = 0,$$

so that $v_j \in V_{\lambda_j}$.* Hence $V = \sum V_{\lambda_j}$.

*Note also that $(T-\lambda_j I)^N v_j = 0$

To complete the proof, we must show

$$V_{\lambda_j} \cap \left(\sum_{k \neq j} V_{\lambda_k} \right) = \{0\}.$$

Assume we have $x_j \in V_{\lambda_j}$ & $x_k \in V_{\lambda_k}$ (all k) such that

$$x_j = \sum_{k \neq j} x_k$$

Apply the identity $I = \sum_m s_m(T)\, q_m(T)$ to both sides. Now $q_m(T) x_j = 0$ if $m \neq j$ so that $\sum_{k \neq j} x_k = x_j = s_j(T) q_j(T) x_j =$ $s_j(T)\, q_j(T) \sum_{k \neq j} x_k$. Now

$(t-\lambda_k)^N$ is a factor of $q_j(t)$ if $k \neq j$, so

$q_j(T)x_k = 0$ (recall that $(T-\lambda_k I)^N x_k = 0$ by the choice of N on page 4A). It follows that

$$g_j(T)q_j(T)\sum x_k = 0,$$

which in turn implies that $x_j = 0$, so that

$$V_{\lambda_j} \cap \Big(\sum_{k\neq j} V_{\lambda_k}\Big) = \{0\}.$$

Finally to prove the decomposition $v = \sum v_j$ is unique, suppose we also have $\sum v_j' = v$. Then

$$\sum v_j = \sum v_j' \Rightarrow \text{ for each } i,$$

$$v_i - v_i' = \sum_{k\neq i} v_k' - v_k,$$

and the preceding argument shows that $0 = v_i - v_i'$, or equivalently $v_i = v_i'$. $\blacksquare$