# 8E. Uniqueness of the Jordan form

In order to prove uniqueness, it is enough to show that certain numerical quantities associated to a Jordan form for a linear transformation $T : V \to V$ are completely determined by properties of $T$.

Primary decomposition. The number of $\lambda$'s down the diagonal of a Jordan form matrix is equal to the dimension of $V_\lambda$.

Because of this, it is enough to consider cases where $V = V_\lambda$ for some $\lambda$.

Rational decomposition of nilpotent transformations

Say $N$ is nilpotent and $\dim V = n$, so $T = \lambda I + N$. Let $b_k$ be the number of $k \times k$ elementary Jordan blocks in some Jordan form. We need to show that these numbers are completely determined by $T$.

Let $d_k = \dim(\text{Kernel } N^k)$ for $1 \le k \le n$.
We shall show that the numbers $b_k$ are completely determined by the numbers $d_m$.

One can check directly that

$$d_m = b_1 + 2b_2 + \dots + (m-1)\,b_{m-1} + m\,(b_m + \dots + b_n).$$

If we set $d_0 = 0$, then

$$d_m - d_{m-1} = b_m + \dots + b_n.$$

so that the sums $b_m + \dots + b_n$ $(1 \le m \le n)$ are determined by the dimensions $d_k$.
We can restate this as

$$b_n = d_n - d_{n-1} \qquad \text{etc}$$
$$b_{n-1} + b_n = d_{n-1} - d_{n-2}$$

and by induction it will follow that each $b_k$ is determined by the dimensions $d_i$.