EXPANSIONS BY MINORS

Recall the formula for expansion by minors along row number k

$A = (a_{ij})$ is $n \times n$

$\det A = \sum_j a_{kj} M_{kj}$ where M_{kj} is $(-1)^{k+j}$ times the determinant of the $(n-1)\times(n-1)$ submatrix formed by deleting row k and column j.

This is particularly useful if some row has several zeros in it.

EXAMPLE.

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

The first row has two zeros, so we shall let $k=1$.

$$\det A = \underbrace{2}_{\text{sign}=(-1)^{1+1}=1} \overset{j=1}{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}} \underbrace{-}_{\text{sign}=(-1)^{1+2}=-1} \overset{j=2}{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}}$$

Evaluate the 3×3 determinants by the standard elementary rule:

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} \begin{matrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{matrix} = 8+0+0-0-2-2 = 4$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{matrix} = 4+0+0-0-0-1 = 3$$

So $\det A = 2 \cdot 4 - 1 \cdot 3 = \boxed{5.}$

ANOTHER EXAMPLE.

$$A = \begin{pmatrix} 2 & 0 & -1 & 3 \\ 0 & -1 & 0 & 4 \\ 3 & -1 & 0 & 5 \\ -1 & 2 & 2 & 0 \end{pmatrix}$$

Expand by minors along row 2.

$$\det A = -\begin{vmatrix} 2 & 1 & -3 \\ 3 & 0 & 5 \\ -1 & 2 & 0 \end{vmatrix} + 4 \begin{vmatrix} 2 & 0 & 1 \\ 3 & -1 & 0 \\ -1 & 2 & 2 \end{vmatrix}$$

Once again evaluate the minors.

$$\begin{vmatrix} 2 & 1 & -3 \\ 3 & 0 & 5 \\ -1 & 2 & 0 \end{vmatrix} = 0-5-18-0-0-20 = -43.$$

$$\begin{vmatrix} 2 & 0 & 1 \\ 3 & -1 & 0 \\ -1 & 2 & 2 \end{vmatrix} = 4+6+0-0-0-1 = 1.$$

Therefore $\det A = (-1)(-43) + 4 \cdot 1 =$

$$43 + 4 = \boxed{47}$$

THIRD EXAMPLE.

Expansion by minors is not always the simplest way to evaluate a determinant.

Here is an example using row operations.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} = A.$$

If we subtract the first row from the other rows, we obtain the following matrix, and its determinant equals the determinant of the original matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Now subtract the second row from the third and fourth. This yields another matrix whose determinant is $\det A$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Now subtract the third row from the fourth. Again the determinant is unchanged and we get the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix is triangular, and hence its determinant is the product of the diagonal entries. The latter product equals

$$\boxed{1.}$$