Solutions to Exercises 5C

X1. The ~~first~~ main thing to show is that if A and B are upper triangular matrices, then the diagonal entries of AB are given by $a_{ii}b_{ii}$. — If we know this, then it follows that the diagonal entries of A^k are a_{ii}^k and hence the diagonal entries of $\sum_j c_j A^j$ are $\sum c_j a_{ii}^j$. all j's.

By the triangularity hypothesis $a_{pq} = b_{pq} = 0$ if $p > q$. Hence

$$(AB)_{ii} = \sum_m a_{im} b_{mi}.$$

Now $a_{im} = 0$ if $i > m$ and $b_{mi} = 0$ if $m > i$, so the only nonzero term in the summation occurs when $m = i$; in other words, the sum collapses to $a_{ii}b_{ii}$.

X2. The statement in brackets is true because A does have a basis of eigenvectors, and their eigenvalues are $a_{11}, \ldots, a_{nn}$. (this is true because the eigenvalues are distinct). Now if $Av = cv$, it follows that $A^k v = c^k v$ and $q(A)v = q(c)v$ for every polynomial $q(t)$. Hence if $p(t) = \prod (t - a_{ii})$, we have, for each j, $p(A)v_j = p(c)v_j = \prod (c - a_{ii}) v$. Since $c_j = a_{kk}$ for some k, the product is zero, and hence $p(A)v_j = 0$ for all j.