Solutions for `exercises5A.pdf`

1. Suppose that $z = W_1 + W_2$ and write $z = x_1 + x_2$ where $x_i \in W_i$. Since each W_i is T-invariant we have $T(x_1) \in W_1$ and $T(x_2) \in W_2$. Therefore

$$T(z) \quad = \quad T(x_1 + x_2) \quad = \quad T(x_1) + T(x_2)$$

lies in $W_1 + W_2$, and hence the latter is T-invariant.■

2. First, follow the hint in `exercises5A.pdf`; we want to show that if $T(v) = cv$ then $T(S(v)) = cS(v)$. But $ST = TS$ implies

$$TS(v) \quad = \quad ST(v) \quad = \quad S(cv) \quad = \quad cS(v)$$

so the claim is true. Similarly, if $S(v) = dv$ then $S(T(v)) = dT(v)$.

Next, follow the hint in `aabUpdate02.132.s18.pdf`. Let V_c be the eigenspace of T corresponding to the eigenvalue T. Then $y \in V_c$ can be written as a sum of eigenvectors $x_1 + ... + x_k$ for S such that the associated eigenvalues (with respect to S) are distinct. We want to prove by induction that each x_j lies in V_c.

If $k = 1$ there is nothing to prove. Suppose that we know the statement is true for $k - 1 \geq 1$. Let d_j is the associated eigenvalue for x_j with respect to S. Then by the first paragraph we know that $S(y) = d_1 x_1 + ... + d_k x_k$ lies in V_c. On the other hand, we also know that $d_k y = d_k x_1 + ... + d_k x_k$ lies in V_c. If we now subtract the second vector from the first, we find that

$$(d_1 - d_k)x_1 + ... + (d_{k-1} - d_k)x_{k-1} \quad \in \quad V_c \ .$$

Since the eigenvalues d_j are distinct, it follows that the coefficieints $d_j - d_k$ are all nonzero and hence the summands are eigenvectors for S with distinct eigenvalues. We can now apply the induction hypothesis to conclude that each $(d_j - d_k)x_j$ is in V_c, and since the coefficients are all nonzero we can also conclude that each x_j lies in V_c. Therefore it follows that

$$x_k \quad = \quad x \ - \sum_{j<k} x_j$$

also belongs to V_c, completing the proof of the inductive step.

Finally, the preceding implies that the intersections of the eigenspaces $V_{c,T}$ for T and $V_{d,S}$ for S give a spanning set for V; note that these subspaces are invariant under both S and T, with T sending a vector $z \in V_{c,T} \cap V_{d,S}$ to cz abd S sending z to dz. Furthermore, if $(c, d) \neq (c', d')$, the usual arguments show that the intersection of the subspaces $V_{c,T} \cap V_{d,S}$ and $V_{c',T} \cap V_{d',S}$ is always empty. Therefore V is a direct sum of these intersections, and it follows that we have simultaneously diagonalized S and T.■

3. The eigenvalues of the triangular matrix

$$A \quad = \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

are 1, 2 and 3. To find the associated eigenvectors we need to find the respective null spaces of the matrices $A - I$, $A - 2I$ and $A - 3I$:

$$A - I = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}, \qquad A - 2I = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A - 3I = \begin{pmatrix} -2 & 2 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Direct computation shows that each of these null spaces is 1-dimensional, and spanning vectors are given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad A - 3I = \begin{pmatrix} 9 \\ 6 \\ 2 \end{pmatrix}$$

respectively. All nonzero multiples of these vectors are also eigenvectors.■

4. Once again, follow the hint. One renumbering which works is $y_j = x_{n-j}$ (in other words, writing the list of basis vectors in the reverse order).■