Solutions for `exercises8A.pdf`

1. Follow the hint and compute ST. Since S is a polynomial in T we have $ST = TS$.

We are given that $p(t) = \sum a_k t^k$ be a polynomial of least degree m such that $p(T) = 0$, so that $a_m \neq 0$, and assume that the constant term a_0 is also nonzero. If $S = \sum_1^m a_0^{-1} a_k T^{k-1}$, then

$$ST = \sum_1^m a_0^{-1} a_k T^k = a_0^{-1} (p(T) - a_0 I) = -I$$

and hence if $S_1 = -S$ then $S_1 T = I$. Now S_1 is a polynomial in T, and therefore we also have $TS_1 = S_1 T$; hence $TS_1 = I$ and consequently S_1, which is a polynomial in T, is equal to T^{-1}.■

2. The most transparent way to work this problem is to use determinants, which are introduced in Section 10B, so this problem can be skipped until we reach that point in the course. The solution given below will use determinants, including the generalization of Exercise 3 in `exercisesTenA.pdf` which is mentioned in the latter.

Suppose first that the eigenvalues are distinct; let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues, and let $v_1, \cdots, v_n$ be eigenvectors associated to these eigenvalues. If $w = \sum v_j$, we claim that the vectors $T^k w$, where $0 \leq w \leq n-1$, form a basis for V.

The first step is to form an $n \times n$ matrix A such that the j^{th} column of A gives the corrdinates of the vectors $T^{j-1}w$ with respect to the basis of eigenvalues $v_1, \cdots, v_n$. By our assumptions $a_{i,1} = 1$ for all i because $w = \sum v_i$. Likewise, we have $T^j w = \sum_{\lambda_i}^j v_i$, and this means that $a_{i,j} = \lambda_i^{j-1}$. Therefore A is the transpose of the $n \times n$ Vandermonde matrix mentioned in the citation from the first paragraph of this solution.

The cited reference for the general Vandermonde matrix shows that the determinants of A and its transpose are given by

$$\prod_{i<j} (\lambda_j - \lambda_i)$$

so if the eigenvalues are distinct this determinant is nonzero. This means that A is an invertible matrix and hence the vectors $T^k w$, where $0 \leq w \leq n-1$, form a basis for V.■

Conversely, suppose there is a basis of eigenvectors but the eigenvalues are not distinct. One more $v_1, \cdots, v_n$ be a basis of eigenvectors, and let $\lambda_1, \cdots, \lambda_n$ be eigenvalues associated to these eigenvectors. We might as well number everything so that $\lambda_1 = \cdots = \lambda_k$ for some $k > 1$ since the eigenvalues are not distinct.

Set V_1 equal to the span of $v_1, \cdots, v_k$ where k is given as in the preceding sentence, and define a linear transformation $P : V \to V_1$ sending v_j to itself if $j \leq k$ and to 0 otherwise. By construction P is onto.

Suppose that there is some vector $x \in V$ such that the vectors $T^k x$ span V, where $0 \leq k \leq m$ for some m. Then the vectors $PT^k x$ also span V_1, and hence there is some $y \in V_1$ such that the vectors $T^k y$ span V_1. But if $w \in V_1$ then $T^k w = \lambda_1^k w$, and hence for each nonzero $w \in V_1$ the vectors $T^k w$ span a 1-dimensional subspace. By assumption $\dim V_1 \geq 2$, so we have a contradiction. The source of this contradiction is our assumption that for some $x \in V$ the vectors $T^k x$ span V, so this is impossible and there is no vector $x \in V$ such that the vectors $T^k x$ span V.■

3. Once again follow the hint. If $\mathbf{e}_1$ is the first standard unit vector, then the vectors $P^j\mathbf{e}_1$ (where $0 \leq j \leq 2$) form a basis because $P\mathbf{e}_1 = \mathbf{e}_2$ and $P^2\mathbf{e}_1 = P\mathbf{e}_2 = \mathbf{e}_3$. But we also have

$$P^3\mathbf{e}_1 \quad = \quad P^2\mathbf{e}_2 \quad = \quad P\mathbf{e}_3 \quad = \quad -\,a\mathbf{e}_1 \;-\; b\mathbf{e}_2 \;-\; c\mathbf{e}_3 \quad =$$

$$-aI\mathbf{e}_1 \;-\; bP\mathbf{e}_1 \;-\; cP^2\mathbf{e}_1$$

which implies that $(aI+bP+cP^2+P^3)\mathbf{e}_1 = 0$ and hence $\mathbf{e}_1$ lies in the kernel of $(aI+bP+cP^2+P^3)$. This implies similar conclusions for $\mathbf{e}_2$ and $\mathbf{e}_3$, for if $j = 2$ or 3 then

$$(aI + bP + cP^2 + P^3)\mathbf{e}_j \quad = \quad (aI + bP + cP^2 + P^3)P^j\mathbf{e}_1 \quad =$$

$$P^j(aI + bP + cP^2 + P^3)\mathbf{e}_1 \quad = \quad P^j0 \quad = \quad 0$$

so that $aI + bP + cP^2 + P^3$ takes every vector in a basis to zero and hence must be the zero linear transformation.∎