Solutions to Exercises 10

10 A. The trace of a matrix

X1. If $B = A^*$ then $b_{jk} = \overline{a_{kj}}$, so
$\text{trace}(B) = \sum b_{kk} = \sum \overline{a_{kk}} = \overline{\sum a_{kk}} = \overline{\text{trace}(A)}$ [complex conjugate].

X2. (a) If A is real-symmetric then $a_{ji} = -a_{ij}$. In particular $a_{ii} = -a_{ii}$ all i, so each $a_{ii} = 0$. Hence $\text{trace}(A) = \sum_i a_{ii} = \sum_i 0 = 0$.

(b) Put a nilpotent matrix A into Jordan form N, so that $N = P^{-1}AP$ and $\text{trace}(A) = \text{trace}(N)$. By construction of the Jordan form, the entries b_{ij} of N satisfy $b_{ij} = 0$ unless $j = i+1$ (and maybe ~~the~~ some or all entries are zero in ~~some of~~ these cases too). Hence $\text{trace}(A) = \text{trace}(N) = \sum_i b_{ii} = \sum_i 0 = 0$.

EXTRA PROBLEM. Give examples to show that trace AB is not necessarily equal to (trace A)(trace B).

Solution. It is natural to search for examples which are as simple as possible.

Let's try diagonal matrices with entries a_{ii}, b_{ii}.

Then trace $AB = \sum_i a_{ii} b_{ii}$ while

trace A trace B $= (\sum_i a_{ii})(\sum_j b_{jj})$. It seems clear that these should be different if the diagonal entries are all positive.

In formal terms, the simplest choice is to take $A = B = I$. If these matrices are $n \times n$, then

$$\text{trace}(AB) = \text{trace}(I) = n$$

$$\text{trace}(A)\,\text{trace}(B) = \text{trace}(I)^2 = n^2$$

and if $n \geq 2$ we have the desired examples because $n^2 > n$. Note that trace(AB) = trace(A) trace(B) does hold if A, B are 1×1 (why?).

10B Determinants

X1 We know that $(1 \dots k)$ is the composite of $(1 \dots k-1)$ followed by $(k-1\ k)$. Therefore

$\operatorname{sgn}(1 \dots k) = -\operatorname{sgn}(1 \dots k-1)$. Now $\operatorname{sgn}(12) = -1$,

and therefore we can prove that

$$\operatorname{sgn}(1, \dots, k) = (-1)^{k-1} \text{ by induction.}$$

ALTERNATE SOLUTION. Let $\sigma = (1 \dots k)$. Then $\sigma^k = 1$, so $1 = \operatorname{sgn} \text{identity} = (\operatorname{sgn}\sigma)^{k-1}$

Suppose now that k is even, so $(k-1)$ is odd. Then the right hand side is $\operatorname{sgn}\sigma$ because $(\pm 1)^{odd} = \pm 1$.

But $(\operatorname{sgn}\sigma)^{k-1} = 1$, so $\operatorname{sgn}\sigma = +1$ if k is even.

Suppose now that k is odd, so $k+1$ is even. Then as above we have

$$\left.\begin{aligned} \sigma &= (1 \dots k) \\ \tau &= (k\ k+1) \end{aligned}\right\} \Rightarrow \tau\sigma = (1 \dots k-1)$$

and $1 = \operatorname{sgn} \tau\sigma = \operatorname{sgn}\sigma \operatorname{sgn}\tau = (\operatorname{sgn}\sigma)(-1)$, so

$\operatorname{sgn}\sigma = -1$.

X2 See what happens when we perform row ops.

$$\begin{pmatrix} 1&1&1&1&1\\ 0&1&1&1&1\\ 1&0&1&1&1\\ 1&1&0&1&1\\ 1&1&1&0&1 \end{pmatrix} \xrightarrow[\Delta = 1]{\text{Subtract row 2 from row 1}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&1&1&1\\ 1&0&1&1&1\\ 1&1&0&1&1\\ 1&1&1&0&1 \end{pmatrix}$$

$$\xrightarrow[\Delta = 1 \text{ for each operation}]{\text{subtract row 1 from rows 3,4,5}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&1&1&1\\ 0&0&1&1&1\\ 0&1&0&1&1\\ 0&1&1&0&1 \end{pmatrix} \xrightarrow[\Delta = 1]{\text{Subtract row 3 from row 2}}$$

$$\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&1&1\\ 0&1&0&1&1\\ 0&1&1&0&1 \end{pmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{subtract row 2 from rows 4,5}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&1&1\\ 0&0&0&1&1\\ 0&0&1&0&1 \end{pmatrix}$$

$$\xrightarrow[\Delta = 1]{\text{subtract 4th from 3rd}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&1\\ 0&0&1&0&1 \end{pmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{subtract 3rd from 5th, then 5th from 4th}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$$

Hence the original matrix is invertible and its determinant is the product of the Δ's, which is 1.

X3. One effective but inelegant way to solve this is to write out det A and the polynomial in the entries described above as the first step. Then one can rewrite the determinant and square in simplified terms to check whether the expressions are equal.

The right hand side is the easiest since
$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc.$$
This yields the following:
$$(a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24})^2 =$$
$$a_{12}^2 a_{34}^2 + a_{14}^2 a_{23}^2 + a_{13}^2 a_{24}^2$$
$$+ 2a_{12}a_{14}a_{23}a_{34} - 2a_{12}a_{13}a_{24}a_{34} - 2a_{13}a_{14}a_{23}a_{24}.$$

Now we need to find det
$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$$

The matrix has this form since ${}^{T}A = -A$.

We can do this by expansion by minors along the first row, and here is what we get:

$\det A =$

Note that $a_{11} = 0$ so the first term vanishes

$$-a_{12}\begin{vmatrix} -a_{12} & a_{23} & a_{24} \\ -a_{13} & 0 & a_{34} \\ -a_{14} & -a_{34} & 0 \end{vmatrix} + a_{13}\begin{vmatrix} -a_{12} & 0 & a_{24} \\ -a_{13} & -a_{23} & a_{34} \\ -a_{14} & -a_{24} & 0 \end{vmatrix}$$

$$-a_{14}\begin{vmatrix} -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \\ -a_{14} & -a_{24} & -a_{34} \end{vmatrix}.$$

Now we have three 3×3 determinants to expand; the placements of the zeros implies that in each case there are only three nonzero terms that appear in the usual formula for 3×3 determinants.

Thus det A is equal to

$$-a_{12}\Big(\overset{\text{half of } 5}{a_{13}a_{24}a_{34}} - \overset{1}{a_{12}a_{34}^2} - \overset{\text{half of } 4}{a_{14}a_{23}a_{34}}\Big)$$

$$+a_{13}\Big(\overset{3}{a_{13}a_{24}^2} - \overset{\text{half of } 6}{a_{14}a_{23}a_{24}} - \overset{\text{half of } 5}{a_{12}a_{24}a_{34}}\Big)$$

$$-a_{14}\Big(\overset{\text{half of } 4}{-a_{12}a_{23}a_{34}} + \overset{\text{half of } 6}{a_{13}a_{23}a_{24}} - \overset{2}{a_{14}a_{23}^2}\Big)$$

Extreme care is needed to get all the signs right!

The comments in green show that the terms in this expression math 1 to 6 [in the written order] for the previous one.

<u>Note</u> It is not difficult to see that if A is skew-symmetric and $n \times n$ then $\det A = 0$ if n is odd and $\det A \geq 0$ if n is even.

[~~Use $\det A = \det {}^{T}A = \det(-A) = (-1)^n$~~

If n is odd, then $\det A = \det {}^{T}A = \det(-A) = (-1)^n \det A = -\det A$, so $\det A = 0$, while if n is even then iA is Hermitian $\Rightarrow$ all nonzero eigenvalues of A are purely imaginary, and they come in conjugate pairs $\lambda_1, \overline{\lambda_1}, \ldots, \lambda_k, \overline{\lambda_k}$. Therefore $\det A = \prod \lambda_j \overline{\lambda_j} = \prod |\lambda_j|^2 \geq 0$ if n is even. Hence $\det A = y^2$ for some y in this case. It's much less obvious that det A is actually a ~~polynomial~~ square of a polynomial in

${}^{t}A = -A$

the entries of A whenever n is even. Here is a web link to a proof of this result:

http://www.physics.drexel.edu/~tim/open/pfaff/pfaff.pdf

X4. These exercises are designed to be solved using row operations

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{bmatrix} \xrightarrow[\Delta = 2]{\text{mult. 1st row by } \frac{1}{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{bmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{subtr. 1st from 2nd \& 4th}}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 5 \end{bmatrix} \xrightarrow[\Delta = -1]{\text{switch 3rd \& 4th}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 3 & 2 \end{bmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{add 3rd to 1st, } \cancel{\text{4th}} \text{ } 3\times\text{3rd to 4th}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 17 \end{bmatrix} \xrightarrow[\Delta = -1, 17]{\text{mult. 3rd by } -1\text{, 4th by } 1/17} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{subtr. mults of 4th from other 3}} I$$

So the matrix is invertible, and its determinant is the product of the Δ's, which is 34.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{subtr 1st from others}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 8 & 8 & 8 & 8 \\ 12 & 12 & 12 & 12 \end{bmatrix} \xrightarrow[\Delta = 4, 8, 12]{\text{mult. 2nd, 3rd, 4th by } \frac{1}{4}, \frac{1}{8}, \frac{1}{12}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

We can stop here (almost) since the last three rows are equal, and hence the determinant is a multiple of 0, which means it $\boxed{\text{IS}}$ zero.

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$$\begin{bmatrix} 3 & 1 & 3 & 0 \\ 3 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 6 & 3 & 4 & 5 \end{bmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{subtr. 1st from 2nd, 4th}} \begin{bmatrix} 3 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & -2 & 5 \end{bmatrix} \xrightarrow[\Delta = -1]{\text{switch 2nd + 4th}}$$

$$\begin{bmatrix} 3 & 1 & 3 & 0 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{subtract mults of 4th from 2nd + 3rd}} \begin{bmatrix} 3 & 1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\Delta = 2]{\text{mult 3rd by } \frac{1}{2}}$$

$$\begin{bmatrix} 3 & 1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{subtr. mults of 3rd from 1st + 2nd}} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\Delta = 1]{\text{subtr. 2nd from 1st}}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\Delta = 3]{\text{mult 1st by } 1/3} I$$

Hence this matrix is invertible, and its determinant is the product

$$(-1) \cdot 2 \cdot 3 = -6.$$

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$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{each } \Delta = -1]{\substack{\text{switch} \\ 5^{th} + 1^{st} \\ 4^{th} + 2^{nd}}} I.$$

Hence the matrix is invertible and its determinant is $(-1)(-1) = 1$.

IMPORTANT NOTE. These sequences of operations are somewhat opportunistic, designed to get the conclusion quickly with a minimum of fractions. In general, if there is no evident opportunistic path then it is best to do row reduction systematically as in Gaussian elimination.

X5. Here is the crucial observation.

CLAIM: If A is an $m \times n$ matrix and it contains a $q \times q$ submatrix with nonvanishing determinant, then the rank of A is at least q.

Proof that the claim solves the exercise. Suppose that A has ~~an $r \times r$ matrix with nonzero~~ rank r.

Then A has no $(r+1)\times(r+1)$ sub-matrices with nonvanishing determinant, for the existence of such a submatrix would imply that $\operatorname{rank}(A) = r+1 > r$. Now suppose that the rank of A is r. Then there are r rows $A_{i_1}, \dots, A_{i_r}$ $(i_1 < \dots < i_r)$ which are linearly independent. Let B be the $r \times n$ matrix formed from these rows. Since column rank B = row rank $B = r$, there are columns $j_1 < \dots < j_r$ in B such that the submatrix C formed by these columns is an $r \times r$ submatrix with rank r. Hence C is an invertible $r \times r$ submatrix of A such that $\det C \neq 0$. ∎

<u>Proof of the claim</u> Let $k_1 < \dots < k_q$, where $q < p$, and let $P : \mathbb{R}^p \to \mathbb{R}^q$ send $(x_1, \dots, x_p)$ to $(x_{k(1)}, \dots, x_{k(q)})$. If $v_1, \dots, v_r$ are vectors in $\mathbb{R}^p$ and $P(v_1), \dots, P(v_r)$ are linearly indep.,

then so are $v_1, \ldots, v_r$; this is true because

$$\sum^r c_j v_j = 0 \Rightarrow 0 = P\left(\sum^r c_j v_j\right) = \sum^r c_j P v_j = 0 \Rightarrow c_j = 0, \text{ all } j.$$

We use this to prove the claim as follows: Suppose $\det C \neq 0$ where C is obtained from A by deleting all rows except $i_1 < \ldots < i_q$ and all columns except $j_1, \ldots, j_q$. Then the rows of C are linearly independent, and hence by the rows $i_1, \ldots, i_q$ in A are also linearly independent. But this means that the (row) rank of A is at least q. ∎

<u>X6.</u> More practice as in X4.

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 4 & 2 & 1 & 5 \\ -1 & 5 & 2 & 6 \\ 4 & 2 & 1 & 5 \end{pmatrix}$$

Two rows are equal, so the determinant $= 0$.

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$$\begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{Switch 1st + 3rd, 2nd + 4th}} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(continued on the next page).

We can now use the block determinant rule $\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A \cdot \det C$ to see that the determinant of the 4×4 matrix is

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \cdot 3 = 9.$$

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$$\begin{pmatrix} 3 & 6 & 9 & 12 \\ 1 & 2 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{pmatrix} \xrightarrow[\Delta = 1]{\text{subtr. 1st from 3rd}} \begin{pmatrix} 3 & 6 & 9 & 12 \\ 1 & 2 & 2 & 1 \\ 0 & -1 & -7 & -11 \\ 0 & 2 & 4 & 2 \end{pmatrix} \xrightarrow[\Delta = 3]{\text{mult. 1st by } 1/3}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \\ 0 & -1 & -7 & -11 \\ 0 & 2 & 4 & 2 \end{pmatrix} \xrightarrow[\Delta = 2]{\text{mult. 4th by } 1/2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \\ 0 & -1 & -7 & 11 \\ 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow[\Delta = 1]{\text{subtr. 1st from 2nd}}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -3 \\ 0 & -1 & 7 & 11 \\ 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow[\text{each } \Delta = 1]{\text{add mults. of 4th to 1st + 3rd}} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 9 & 12 \\ 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow[\Delta = 1]{\text{cyclically permute 2nd, 3rd, 4th}}$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 9 & 12 \end{pmatrix}.$$

By the block determinant rule, this matrix has determinant -39, so $-39 = \dfrac{\det A}{2 \cdot 3} \Rightarrow$

$\det A = -39 \cdot 6 = -234.$