Solutions for aabUpdate07.132.w17.pdf

1. (a) Write $v \in V$ as $x+y$ where $x \in W$ and $y \in W^\perp$. Then $Ev = x$. Now $x = x + 0$ where $0 \in W^\perp$, so $Ex = x$. Hence $E^2 v = Ex = x = Ev$, so that $E^2 = E$.

To see self-adjointness, also write $w = u' + v'$ where $u' \in W$ and $v' \in W^\perp$. Then

$$\langle Ev, w\rangle = \langle x, u'+v'\rangle \underset{x \perp v'}{=} \langle x, u'\rangle$$

$$= \langle x, Ew\rangle \underset{y \perp Ew}{=} \langle x+y, Ew\rangle = \langle v, Ew\rangle$$

which shows that $E^* = E$. ∎

(b) Let $x \in V$. Then $E_1 E_2 x \in W_1$ and $E_2 E_1 x \in W_2$; since $E_1 E_2 = E_2 E_1$, it follows that $E_1 E_2 x = E_2 E_1 x \in W_1 \cap W_2$.

Suppose that $x \in W_1 \cap W_2$. Then

$E_1 x = E_2 x = x$, and

$$E_1 E_2 x = E_1 x = x.$$

Now suppose $y \in (W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$.

Write $y = y_1 + y_2$, where $y_i \in W_i^\perp$. Then

$$E_1 E_2 y = E_1 E_2 y_1 + E_1 E_2 y_2 \underset{(E_1E_2 = E_2E_1)}{=}$$

$$E_2 \underbrace{E_1 y_1}_{=0} + E_1 \underbrace{E_2 y_2}_{=0} =$$

$$0 + 0 = 0.$$

Hence $(W_1 \cap W_2)^\perp$ is the kernel of $E_1 E_2$. ∎

2. $(\Rightarrow)$ Suppose T is invertible. Let $S = T^{-1}$. Then $ST = I = TS \Rightarrow$

$$T^* S^* = (ST)^* = \underset{\underset{I}{\|}}{I^*} = (TS)^* = S^* T^*.$$

Therefore $I^* = I \Rightarrow S^* = (T^*)^{-1}$.

$(\Leftarrow)$ If T^* is invertible, go through the preceding to show $T^{**} = T$ is invertible. ∎

3. (a) Suppose T_1 + T_2 are conformal and write $T_i = r_i S_i$ where S_i is orthogonal. $r_i > 0$

Then $T_1 T_2 = r_1 r_2 S_1 S_2$; since $S_1 S_2$ is orthogonal, $T_1 T_2$ is conformal.

Likewise $(r_1 S_1)^{-1} = r_1^{-1} S_1^{-1}$, which is conformal since $r_1 > 0$ and S_1^{-1} is orthogonal. ∎

(b) If $T = rS$ is conformal as above then $T^* = rS^*$, and since $S^* = S^{-1}$ we have $T^*T = r^2 S^* S = r^2 I = r^2 SS^* = TT^*$. ∎

4. Let $A = \begin{pmatrix} 2 & 4 \\ 4 & k \end{pmatrix}$. If $\det A = 2k - 16$ is positive, then A is positive definite, while if $\det A < 0$ then A is neither positive definite nor positive semidefinite because it has a negative eigenvalue.

Now $\det A > 0$ if $k > 8$ so A is pos def
$\det A < 0$ if $k < 8$ so A is neither

in such cases.

What if $\det A = 0$, so that $k = 8$?
$\det A = 0 \Rightarrow \operatorname{Ker} A \neq \{0\} \Rightarrow 0$ is an eigenvalue. To get the other eigenvalue, we can use determinants or the formula
$\lambda_1 + \lambda_2 =$ sum of diagonal entries $= 10$.
Since (say) $\lambda_1 = 0$, we must have $\lambda_2 = 10$, which means A is positive semidefinite if $k = 8$. ∎

5. Let $w_1, \ldots, w_n$ be an orthonormal basis of eigenvectors for A, with eigenvalues ordered so that $\lambda_1 \leq \ldots \leq \lambda_n$. Then $x = \sum c_j w_j \Rightarrow$
$\langle Ax, x\rangle = \sum \lambda_j c_j w_j^2$. We get a lower estimate by replacing the λ_j with λ_1 $= \alpha$)

and we get an upper estimate if we replace the λ_j with $\lambda_n = \beta$.

More precisely

$$|x|^2 \alpha \le \langle Ax, x \rangle \le \beta |x|^2$$

and if $x \neq 0$ we can divide by $|x|^2$ to obtain

$$\alpha \le \frac{\langle Ax, x \rangle}{|x|^2} \le \beta. \blacksquare$$

6. It is only necessary to compute

$$\frac{\langle Ax, x \rangle}{|x|^2} \quad \text{for } A \text{ and } x \text{ as}$$

in the problem, obtaining a value of

$$\frac{14}{3}$$

as a lower bound for the largest eigenvalue. $\blacksquare$