Mathematics 132, Spring 2018, Take Home Assignment

Answer Key

1. Show that the 3×3 matrix

$$\begin{pmatrix} 2 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{pmatrix}$$

has a basis of eigenvectors if and only if $b = -ac$.

SOLUTION

Let A be the displayed matrix. Then A has a basis of eigenvectors if and only if the kernel (null space) of $A - 2I$ has rank 1 (the eigenspace is either 1- or 2-dimensional depending upon whether the rank of $A - 2I$ is 2 or 1). Hence it is enough to see when the matrix

$$A \ - \ 2I \ = \ \begin{pmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{pmatrix}$$

has rank 1. This happens if and only if the rows are linearly independent, which is equivalent to saying that the first row is a multiple of the second (which we know is nonzero because the second entry is -1. In other words, the condition is that $(0, a, b)$ is a scalar multiple of $((0, -1, c)$. This happens if and only if $b = -ac$.■

2. Give an example of a 3×3 matrix A such that A is in Jordan form but A^2 is not.

SOLUTION

Thhe simplest nontrivial Jordan form matrices are those with zeros down the diagonal, so we shall consider such examples. If we do this for

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then we have

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and hence A^2 is not in Jordan form.■

3. Let A be the 2×2 matrix

$$\begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix} .$$

Find a basis of (real) eigenvectors for A, say $\{v_1, v_2\}$ such that the first coordinate of each vector is positive, and find the cosine of the angle $\angle v_1 0 v_2$.

SOLUTION

The eigenvalues of A are the roots of the equation $\det(A - tI) = 0$. The latter quadratic equation is equal to $t^2 - 6t + 8 = (t-4)(t-2)$, so the eigenvalues of A are 2 and 4. Their associated eigenvectors generate the null spaces of

$$A - 2I = \begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix} \quad \text{and} \quad A - 4I = \begin{pmatrix} -3 & -1 \\ 3 & 1 \end{pmatrix}$$

respectively. Spanning vectors for these subspaces with positive first coordinates are given by $v_2 = (1, -1)$ and $v_4 = (1, -3)$ or any positive multiples of these.

We can now find the angle between the eigenvectors by the standard formula:

$$\cos \angle v_2 0 v_4 = \frac{\langle v_2, v_4 \rangle}{|v_2| \cdot |v_4|} = \frac{4}{\sqrt{2} \cdot \sqrt{10}} = \frac{2}{\sqrt{5}} .$$

Although the problem did not ask for this information, we note that the angle in question is equal to 26.565051 degrees up to several decimal places.■

NOTE. By the converse (equivalently, the inverse) of the diagonalization theorem for real symmetric matrices, if a real matrix has a basis of real eigenvectors with distinct eigenvalues but is not symmetric, then the eigenvectors in the basis cannot form an orthogonal set.

4. Determine whether the Spectral Theorem applies to each of the following matrices:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3i \\ 2 & 4 & 6 \\ 3i & 6 & 9 \end{pmatrix}$$

SOLUTION

The first matrix satisfies $A^{\mathbf{T}} = -A$, so that A and $A^{\mathbf{T}}$ commute and hence the matrix is normal. Therefore the Spectral Theorem applies.

The second matrix is symmetric, but it is over the complex numbers and it is not self adjoint because the $(1,3)$ and $(3,1)$ entries are not complex conjugates of each other. However, this is not enough to show that the Spectral Theorem fails; we need to show explicitly that A and A^* do not commute. It suffices to find some (p,q) such that the corresponding entries of AA^* and A^*A are not equal. Let's see what happens if we look at the $(1,2)$ entries. For AA^* we get $6+18i$, but for A^*A we get $6-18i$. Therefore the conclusion of the Spectral Theorem is not valid for the second matrix.■

5. Find the determinant of the following 4×4 matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 3 \\ 3 & -4 & 0 & 0 \\ 4 & 3 & 0 & 0 \end{pmatrix}$$

SOLUTION

We want to see the relationship between the determinants of the following two 4×4 matrices in 2×2 block form:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \qquad \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$$

If these determinants are equal, then we can use the fact that the first determinant is $\det A \cdot \det B$ to conclude that the second is also $\det A \cdot \det B = 25 \cdot 4 = 100$.

We can get from the second matrix to the first by a sequence of four adjacent transpositions. First switch 2 and 3, then switch 1 and 2, then switch 3 and 4, and finally switch 2 and 3; if this is hard to grasp mentally, check it using four labeled squares. Since one matrix is obtains from the other by an even number of adjacent transpositions, the rules for determinants imply that the two values are equal.■

Generalization. If A and B are both $n \times n$ matrices, then

$$\det \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} = (-1)^n \cdot \det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} .$$