# EXERCISES FOR CHAPTER 7

### 7A. Linear transformations and inner products

*Exercises in Axler:* 1,3, 4, 7, 12

*Additional exercises:*

**X1.** Let $V$ be a finite dimensional inner product sign over the real or complex numbers, and let $T : V \to V$ be a linear transformation. Prove that $T$ has a decomposition as a sum $S_1 + S_2$ where $S_1$ is self adjoint and $S_2$ is skew adjoint (*i.e.*, $S_2^* = -S_2$). [*Hint:* Consider $\frac{1}{2}(S_1 + S_1^*)$.]

**X2.** Find the eigenvalues and eigenvectors of the following symmetric matrices:

$$\begin{pmatrix} 0 & -2 \\ -2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

**X3.** Suppose that $A$ is a square matrix over the real numbers such that $A$ is skew symmetric. If $i$ is a square root of $-1$, explain why the complex matrix $i\,A$ is Hermitian.

### 7B. The Spectral Theorem

*Exercises in Axler:* 2, 3, 6, 15

*Additional exercises:*

**X1.** Show that the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

is orthogonal and has two real eigenvalues. What are the corresponding eigenvectors?

**X2.** Show that the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is orthogonal and has two nonreal eigenvalues provided $\theta$ is not an integral multiple of $\pi$ (with angle measures expressed in radians). What are the corresponding eigenvectors?

### 7C. Positive operators

*Exercises in Axler:* 1, 2, 5, 6, 9

*Additional exercises:*

**X1.** Determine whether the following matrices are positive definite:

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

**X2.** Let $A$ be a real symmetric matrix. Prove that there is some real number $c > 0$ such that $A + cI$ is positive definite.

**X3.** $(a)$ Define a relation $\leq$ on symmetric $n \times n$ matrices such that $A \geq B$ if and only if $A - B$ is positive semidefinite, and prove that this relation defines a partial ordering on the set of all such matrices. (In other words, $A \geq A$, $A \geq B$ and $B \geq A$ imply $A = B$, $A \geq B$ $B \geq C$ imply $A \geq C$.)

$(b)$ Construct a symmetric $2 \times 2$ matrix $A$ such that neither $A \geq 0$ nor $0 \geq A$ is true.