MORE EXERCISES FOR CHAPTER 8

1. Let V be a finite dimensional vector space, and let $T : V \to V$ be a linear transformation. By the lectures we know that there are nonconstant polynomials $f(t)$ such that $f(T)$ is the zero linear transformation on V.

Let $p(t) = \sum a_k t^k$ be a polynomial of least degree m such that $p(T) = 0$, so that $a_m \neq 0$, and assume that the constant term a_0 is also nonzero. Prove that T is invertible and in fact T^{-1} is a polynomial in T. [*Hint:* Look at $S = \sum_1^m a_0^{-1} a_k T^{k-1}$ and find ST.]

2. Let V be a finite dimensional vector space, and let $T : V \to V$ be a diagonalizable linear transformation. Prove that there is a vector x such that the vectors T^k span V if and only if the eigenvalues are distinct.

3. Let P be the following matrix:

$$\begin{pmatrix} 0 & 0 & -a \\ 1 & 0 & -b \\ 0 & 1 & -c \end{pmatrix}$$

Verify that $aI + bP + cP^2 + P^3 = 0$. [*Hint:* This can be done without computing the powers of P explicitly. If $\mathbf{e}_1$ is the first standard unit vector, why are the vectors $P^j \mathbf{e}_1$ (where $0 \leq j \leq 2$) a basis? Why does $\mathbf{e}_1$ lie in the kernel of $aI + bP + cP^2 + P^3$, and why does this imply a similar conclusion for $\mathbf{e}_2$ and $\mathbf{e}_3$?]