Mathematics 132, Spring 2018, Examination 1

Answer Key

1. [*30 points*] (a) Find the eigenvalues of the following matrix:

$$\begin{pmatrix} 6 & 1 \\ 3 & 1 \end{pmatrix}$$

(b) The eigenvalues of the matrix

$$\begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}$$

are 9 and 2. Find eigenvectors associated to these two eigenvalues.

SOLUTION

(a) The eigenvalues are given by the roots of

$$\det A - tI \quad = \quad \begin{pmatrix} 6-t & 1 \\ 3 & 1-t \end{pmatrix} \quad = \quad 3 \; - \; 7t \; + \; t^2$$

and these roots are equal to

$$\frac{7}{2} \; \pm \; \frac{\sqrt{49-12}}{2} \quad = \quad \frac{7 \, \pm \, \sqrt{37}}{2}$$

by the Quadratic Formula.■

(b) If A is the matrix in the problem, then the eigenvectors associated to 9 and 2 are the null spaces of $A - 9I$ and $A - 2I$ respectively. We now write out these matrices explicitly:

$$A \; - \; 9I \quad = \quad \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix} \; , \qquad A \; - \; 2I \quad = \quad \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$$

The nontrivial solutions for these systems of linear homogeneous equations are given by the nonzero multiples of $(1,1)$ and $(4,-3)$ respectively.■

2. [*20 points*] (a) Give an example of an upper triangular matrix which does not have a basis of eigenvectors.

(b) Let V be a finite dimensional inner product space over the real or complex numbers, let $T : V \to V$ be a linear transformation, and let T^* denote the adjoint transformation. Prove that T^*T is self adjoint.

SOLUTION

(a) The 2×2 matrices of this form are given by

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

where a is an arbitrary scalar and $b \neq 0$. There are also many other examples with 3 or more rows and columns.■

(b) Consider the following chain of identities:

$$(T^*T)^* \quad = \quad T^*T^{**} \quad = \quad T^*T$$

We can pass from the second expression to the third because T^{**} is always equal to T.■

3. [*20 points*] Prove the polarization identity for real inner products (there is a similar but slightly different such identity in the complex case):

$$\langle\, x,\, y\,\rangle \;=\; \tfrac{1}{4}\cdot\left(|x+y|^2 \;-\; |x-y|^2\right)$$

SOLUTION

Since we are dealing with real inner products we always have

$$|v+w|^2 \;=\; \langle\, v+w\,,\, v+w\,\rangle \;=$$

$$\langle\, v\,,\, v\,\rangle \;+\; 2\cdot\langle\, v\,,\, w\,\rangle \;+\; \langle\, w\,,\, w\,\rangle \;=\; |v|^2 \;+\; 2\cdot\langle\, v\,,\, w\,\rangle \;+\; |w|^2\;.$$

If we set $v+w = x\pm y$ in the right hand expression of the problem we obtain the following:

$$\tfrac{1}{4}\cdot\left(|x+y|^2 \;-\; |x-y|^2\right) \;=$$

$$\tfrac{1}{4}\left(|x|^2 \;+\; 2\cdot\langle\, x\,,\, y\,\rangle \;+\; |y|^2\right) \;-\; \tfrac{1}{4}\left(|x|^2 \;-\; 2\cdot\langle\, x\,,\, y\,\rangle \;+\; |y|^2\right)$$

and the latter simplifies to $\frac{1}{4}\cdot 4\langle\, x\,,\, y\,\rangle = \langle\, x\,,\, y\,\rangle$.$\blacksquare$

4. [*30 points*] Let A be the following 3×3 matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Verify that $A^* = A^2$, and compute both A^*A and AA^*.

SOLUTION

We shall merely state the results and not write out the matrix multiplications explicitly. If we carry out the latter we find that

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and therefore $A^2 = A^*$ for this choice of A.

To complete the problem, one computes the products A^*A and AA^*, and it turns out that both are equal to the identity matrix.■

NOTE. These matrix identities reflect the fact that A is a matrix representing the linear transforamtion which permutes the three unit vectors cyclically, sending $\mathbf{e}_1$ to $\mathbf{e}_2$, $\mathbf{e}_2$ to $\mathbf{e}_3$, and $\mathbf{e}_3$ to $\mathbf{e}_1$. It folllows that A^2 represents the results of applying this permutation twice, and since $A^* = A^2$ it follows that $A^3 = A^*A = AA^*$ represents the result of applying it three times. But if we apply the permuation three times we obtain the identity.

We can also see the results without computing anything as follows (but this uses material from Section 7B): Every linear transformation permuting unit vectors is an orthogonal transformation, and therefore $A^* = A^{-1}$. Furthermore, since $A^3 = I$ it follows that $A^2 = A^{-1}$ and consequently $A^2 = A^{-1} = A^*$.