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Mathematics 132, Winter 2017, Examination 3

Answer Key

1. [*30 points*] (a) Let A be an $n \times n$ matrix, and let P be an invertible $n \times n$ matrix. Prove that $\left(PAP^{-1}\right)^3 = PA^3P^{-1}$.

(b) Suppose that N is a nilpotent 4×4 matrix with minimal polynomial z^2. Find all possible Jordan forms for N.

SOLUTION

(a) We have

$$\left(PAP^{-1}\right)^3 = PAP^{-1}PAP^{-1}PAP^{-1}$$

and if we cancel all copies of $P^{-1}P = I$ in this expression we get $PAAAP^{-1} = PA^3P^{-1}$.■

(b) The Jordan form is a block sum of $k_i \times k_i$ elementary nilpotent Jordan submatrices, with at least one 2×2 submatrix since the minimal polynomial is z^2, no larger submatrices, and the sum of the sizes of the matrices equals 4. The only possibilities consistent with these constraints on the sizes are $2 + 2$ and $2 + 1 + 1$.■

2. [*25 points*] Let $\lambda \neq 0$ be a scalar, and let A be the elementary Jordan matrix

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} .$$

Explain why A^2 is not in Jordan form, find the Jordan form for A^2, and justify your answer.

SOLUTION

Write $A = \lambda\, I + N$, where N is the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

Then $A^2 = \lambda^2 I + 2\lambda\, N + N^2$, which equals

$$\begin{pmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix} .$$

This matrix is not in Jordan form because its $(1,3)$ entry is nonzero. To find its Jordan form, we need to write $A^2 = \lambda^2 I + N_2$, where N_2 is the matrix

$$\begin{pmatrix} 0 & 2\lambda & 1 \\ 0 & 0 & 2\lambda \\ 0 & 0 & 0 \end{pmatrix} .$$

This nilpotent matrix satisfies $N_2^2 \neq 0$, so there is a basis x_1, x_2, x_3 such that $N_2 x_1 = x_2$, $N_2 x_2 = x_3$ and $N_2 x_3 = 0$. Therefore the Jordan form of A^2 is equal to

$$\begin{pmatrix} \lambda^2 & 1 & 0 \\ 0 & \lambda^2 & 1 \\ 0 & 0 & \lambda^2 \end{pmatrix} . \blacksquare$$

3. [*20 points*] (a) For some $n \geq 2$, find an $n \times n$ matrix A such that trace $\left(A^{-1}\right) \neq (\operatorname{trace} A)^{-1}$.

(b) Let A be a 6×6 matrix such that $\det A > 0$. Explain why $\det(-A) > 0$ is also true.

SOLUTION

(a) One way to solve this is to focus on diagonal matrices. If we let $A = 2\,I$, then $A^{-1} = \frac{1}{2}\,I$ so that the trace of A is $2n$ but the trace of A^{-1} is $\frac{1}{2}\,n$. This give the desired example because $2n > \frac{1}{2}\,n$ for all positive integers n.■

(b) We have $\det(-A) = (-1)^6 \det A$, where the right hand side is equal to $\det A$ because $(-1)^6 = 1$. Therefore we have $\det(-A) = \det A > 0$.■

4. [*25 points*] Find the determinant of the following 4×4 matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 6 & 5 & 1 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

SOLUTION

Here is one quick way of carrying out the computation: First subtract multiples of the first row from the remaining rows; more precisely, subtract 7, 6 and 3 times this row from the second, third and fourth rows respectively. Then all the entries of the second column are zero except for $a_{1,2} = 1$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 4 & 0 & 2 & 1 \end{pmatrix}$$

These operations leave the determinant unchanged, so the new matrix has the same determinant as the old one. Now subtract 5 and 2 times the second row from the third and fourth rows respectively. This yields the following matrix whose determinant is the same as that of the original matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

Now cyclically permute the matrix via the permutation (1234). The effect of this is to multiply the matrix by the sign of (1234), which is -1, and here is the new matrix:

$$\begin{pmatrix} 4 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

One can now use the formula for the determinant of a triangular matrix to see that the determinant of the last matrix is 4. Now the net effect of the previous row operations was to multiply the determinant by (-1), so the determinant of the original matrix is $4/(-1) \;=\; -4.$■