Mathematics 132, Winter 2021, Examination 1

Answer Key

1. [*25 points*] Suppose that we have an ordered orthonormal basis $\mathbf{B} = \{x_1, ..., x_n\}$ for an inner product space V. Let $\{u_1, \cdots, u_n\}$ be the ordered orthonormal basis obtained from $\mathbf{B}$ by the Gram-Schmidt orthonormalization proces. Prove by induction that $u_k = x_k$ for $k = 1, \cdots, n$.

SOLUTION

Suppose that $k = 1$. In this case $u_1 = |x_1|^{-1}x_1$ and this is equal to x_1 because $|x_1| = 1$ is assumed.

Suppose now that the result is true for $k - 1 > 1$. Then $u_k = |y|^{-1}y$ where

$$y \quad = \quad x_k \; - \; \sum_{j=1}^{k-1} \langle x_k, u_j \rangle \, u_j$$

and this collapses to x_k because $u_j = x_j$ if $j < k$ **(the induction hypothesis)** and $\langle x_k, x_j \rangle = 0$ if $k \neq j$. Hence $y = x_k$, and since $|x_k| = 1$ it follows that $u_k = |x_k|^{-1}x_k = x_k$, which completes the verification of the inductive step.■

2. [*25 points*] Let A, B, C and D be the last four digits of you student identification number in that order. Fid the least squares approximation for the following data:

x =	1	2	3	4	5
y =	1+0.A	2–0.B	3	4+0.C	5–0.D

See the document `.../week02/least-squares-example.pdf` for a solved problem of this type.

SOLUTION

The answer depends upon the numbers A, B, C and D. Questions about the answers in special cases should be addressed to the course instructor.■

3. [*25 points*] Given a complex number $z = a + bi$, show that the map M sending z to

$$M(z) \quad = \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is normal and satisfies $M(z + w) = M(z) + M(w)$ and $M(zw) = M(z)M(w)$. Also show that the eigenvalues are $a \pm bi$.

SOLUTION

First note that we can write $M(z) = aI + bJ$ where J is the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and J satisfies $J^2 = -I$. If $w = c + di$ then we have

$$M(z + w) \;=\; M((a + bi) + (c + di)) \;=\; M((a + c) + (b + d)i) \;=\; (a + c)I + (b + d)J \;=$$

$$(aI + bJ) + (cI + dJ) \;=\; M(z) + M(w)$$

and similarly

$$M(z \cdot w) \;=\; M((a + bi) \cdot (c + di)) \;=\; M((ac - bd) + (ad + bc)i) \;=$$

$$(ac - bd)I + (ad + bc)J \;=\; (aI + bJ)(c + dJ) \;=\; M(z) \cdot M(w)\ .$$

Finally, the same sorts of manipulations also show that $M(z^*) \;=\; M(z)^*$, where the asterisk on the left denotes complex conjugation and the asterisk on the right denotes the transpose (equivalently, adjoint) matrix. This is true because $J* = -J$ implies

$$M(z)^* \;=\; M(a + bi)^* \;=\; (aI + bJ)^* \;=\; aI^* + bJ^* \;=\; aI - bJ^* \;=\; M(a - bi) \;=\; M(z^*)\ .$$

Therefore we have

$$M(z)M(z)^* \quad = \quad M(zz^*) \quad = \quad M(z^*z) \quad = \quad M(z)^*M(z)$$

because $zz^* = z^*z$ in the complex numbers, showing that $M(z)$ is a normal matrix. Therefore $M(z)$ has an orthonormal basis of eigenvectors over the complex numbers, and since $M(z)$ has real entries these eigenvlues form a conjugate pair. They are given by the roots of $0 = (a^2 + b^2) - 2at + t^2 = 0$. By the Quadratic Formula these roots are equal to $a \pm \sqrt{a^2 - (a^2 + b^2)}$, which is just $a \pm bi$.■

4. [*25 points*] Let A be a square matrix over the complex numbers. Prove that each of $A+A^*$, $A-A^*$, and AA^* has an orthonormal basis of eigenvectors, and the eigenvalues are real in first and last cases and imaginary in the second.

SOLUTION

The first and third are the simplest because it is enough to show that the matrices are self-adjoint. These can be shown by the following chains of equations:

$$(A+A^*)^* \;=\; A^*+A^{**} \;=\; A+A^* \;, \quad (AA^*)^* \;=\; A^{**}A^* \;=\; AA^*$$

Since self-adjoint matrices have real eigenvalues, we know that both of the examples have this property.

In contrast, the matrix $A-A^*$ is skew-adjoint; in other words its adjoint is also its negative:

$$(A-A^*)^* \;=\; A^*-A^{**} \;=\; -(A-A^*)$$

This matrix is also normal because $B^* = -B$ implies $BB^* = B(-B) = -B^2 = (-B)B = B^*B$. There are a few ways of showing that the eigenvalues are purely imaginary (for example, one can use the argument in the self-adjoint case as a model), but one can also show this by proving that the matrix $i(A-A^*)$ is self-adjoint and hence has real eigenvalues (for all complex numbers z, the product iz is real if and only if z is purely imaginary). More generally, we claim that if B is skew-adjoint then iB is self-adjoint. This can be derived as follows:

$$(iB)^* \;=\; i^*B^* \;=\; (-i)(-B) \;=\; iB$$

In particular, this implies that $A-A^*$ has an orthonormal basis of eigenvectors with purely imaginary eigenvalues.■