Mathematics 132, Winter 2021, Examination 2

Answer Key

1. [*25 points*] (a) Let A be a real symmetic $n \times n$ matrix with real entries that is positive definite. Prove that there is a minimum value $c^* > 0$ such that $A - c^* I$ is **NOT** positive definite. [*Hint:* Look first at the case where A is diagonal. Why does this shed light on the general case?]

(b) Find c^* when A is the following matrix:

$$\begin{pmatrix} 8 & 3 \\ 3 & 5 \end{pmatrix}$$

SOLUTION

(a) A positive definite real symmetric $n \times n$ matrix has an orthonormal basis of eigenvectors (it is symmetric), and these eigenvalues are all positive if and only if A is positive definite. Therefore, if λ_- is the least (positive) eigenvalue for such a matrix, then $A - cI$ is positive definite if $c < \lambda_-$ and not positive definite if $c \geq \lambda_-$. This means that there is a minimum value c^*, and it is λ_-.■

(b) The eigenvalues of this matrix are the roots of the characteristic polynomial $\det(A - tI$, and for our example this polynomial is $t^2 - 13t + 31 = 0$. The roots of the latter are

$$\frac{13 \pm \sqrt{169 - 124 = 45}}{2}$$

and hence $\lambda_- = c^*$is the smaller of these numbers; namely, $\frac{1}{2}(13 - 3\sqrt{5})$.■

2. [*25 points*] Suppose that A is a 3×3 orthogonal matrix, and consider the normal form as described in `week06/orthog-nform.pdf`. Show that this matrix must have a 2×2 block summand if $A^2 \neq I$. (There is also a converse: If $A^2 = I$ then the normal form is a block sum of 1×1 matrices.)

SOLUTION

We shall prove the contrapositive: If the normal form for A is block sum of 1×1 matrices, then $A^2 = I$.

To see this, let A be a matrix satisfying the block sum condition, and let the normal form be the diagonal orthogonal matrix D such that $P * AP = D$, where P is some orthogonal matrix. Since the diagonal entries of a diagonal orthogonal matrix are ± 1 (orthogonality means that vector lengths are preserved), it follows that $D^2 = I$. Furthermore, if we left multiply D by P and right multiply it by P^*, we see that $A = PDP^*$ and hence

$$A^2 \quad = \quad (PDP^*)^2 \quad = \quad PDP^*PDP^* \quad = \quad PDIDP^*$$

where the final equality holds because $P* = P^{-1}$. But now we have

$$PDIDP^* \quad = \quad PD^2P^* \quad = \quad PIP^* \quad = \quad PIP^* \quad = \quad I$$

and hence $A^2 = I$, which is what we wanted to prove.■

3. [*25 points*] Let $\lambda \neq 0$ be a scalar, and let N be the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so that $\lambda I + N$ is an elementary Jordan matrix. Show that $(\lambda I + N)^3$ is not in Jordan form, and show that its Jordan form is $\lambda^3 I + N$. [*Hint:* What is $(\lambda I + N)^3 - (\lambda^3 I)$?]

SOLUTION

The first step is to compute $(\lambda I + N)^3$. Since λI and N commute, we can do this using the Binomial Theorem:

$$(\lambda I + N)^3 \quad = \quad \lambda^3 I \;+\; 3\lambda^2 N \;+\; 3\lambda N^2 + \; N^3 \quad =$$

$$\begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda & 1 \\ 0 & \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}$$

This matrix is not in Jordan form because its $(1,4)$ entry is nonzero and if C is a matrix in Jordan form we have $c_{i,j} = 0$ if $j \neq i, i+1$ (and some entries in these positions could also be zero).

We know that the Jordan form of $(\lambda I + N)^3$ has the form $\lambda^3 I + P$ where P is a strictly upper triangular matrix which is similar to $Q = (\lambda I + N)^3 - (\lambda^3 I)$. The final assertion about the Jordan form of $(\lambda I + N)^3$ is equivalent to the condition $Q^3 \neq 0$ (for all the other possible Jordan forms P we have $P^3 = 0$), and this condition will hold if we can verify that $Q^3 \mathbf{e}_j \neq \mathbf{0}$ for some $j \in \{1, 2, 3, 4\}$. We can compute these explicitly, and here is what we obtain if $j = 4$:

$$Q^2 \mathbf{e}_4 \quad = \quad Q\left(3\lambda^2 \mathbf{e}_3 + 3\lambda \mathbf{e}_2 + \mathbf{e}_1\right) \quad =$$

$$9\lambda^4 \mathbf{e}_2 + 18\lambda^3 \mathbf{e}_1$$

$$Q^3 \mathbf{e}_4 \quad = \quad Q\left(9\lambda^4 \mathbf{e}_2 + 18\lambda^3 \mathbf{e}_1\right) \quad =$$

$$Q\left(9\lambda^4 \mathbf{e}_2\right) \quad = \quad 27\lambda^6 \mathbf{e}_1$$

The right hand side of the last equation iz zero if and only if $\lambda = 0$, and since $\lambda \neq 0$ we have $Q^3 \neq 0$, and we have already noted that the latter implies the Jordan form of $(\lambda I + N)^3$ is given by $\lambda^3 I + N$.■

4. [*25 points*] (a) Find the eigenvalues for the following real symmetric matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{pmatrix}$$

(b) For each rational eigenvalue α, find a basis for the space of associated eigenvectors.

SOLUTION

(a) The characteristic polynomial $\chi_A(t)$ of this matrix is equal to

$$\begin{vmatrix} 1-t & 1 & 1 \\ 1 & 2-t & 2 \\ 1 & 2 & 4-t \end{vmatrix} \quad = \quad 2 \ - \ 8t \ + \ 7t^2 \ - \ t^3 \ . \blacksquare$$

(b) We need to find the rational roots of the polynomial that we have computed. By Gauss's theorem, we know that every rational root is an integer which evenly divides the constant term, which is 2. Therefore the only possibilities are ± 1 and ± 2. Direct substitution shows that $\chi_A(1) = 0$, $\chi_A(-1) = 18$, $\chi_A(2) = 6$ and $\chi_A(-2) = 54$. Therefore 1 is the only rational root and hence the only rational eigenvalue of A. — In fact, the characteristic polynomial factors as $(1-t)(2-6t+t^2)$ and one can use the Quadratic Formula to show that the quadratic factor has no rational roots and the remaining roots are $3 \pm \sqrt{7}$, both of which are positive (none of this is needed to solve the problem, however).

To find a basis for the associated eigenspace, we need to find the null space of

$$A \ - \ I \ = \ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \ , \quad \text{which is row} - \text{equivlent to} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

The single vector

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

spans this null space and hence spans the eigenspace for the eigenvalue 1.$\blacksquare$

5. [*25 points*] Find the determinant of the matrix displayed below. You may use any valid method to carry out the computation(s).

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \end{pmatrix}$$

SOLUTION

The most efficient way of computing the determinant is by performing row operations on the given matrix to obtain an upper triangular matrix, keeping track of the effects of the operations on the determinants. A first step is to subtract multiples of the first row from the other rows. These operations yield the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Subtracting a multiple of one row from another leaves the determinant unchanged, so the determinant of the new matrix equals the determinant of the original one. Since the determinant of an upper triangular matrix is the product of the diagonal entries, we know that the determinants of both matrices are equal to 1.∎

6. [*25 points*] Let V be an n-dimensional inner product space over the complex numbers, suppose that $T : V \to V$ is a normal operator, and let $n \geq 2$ be an integer. Prove that T^n is also normal.

SOLUTION

One quick way of showing this is to note that T has an orthonormal basis of eigenvectors because it is normal, and the same (othonormal) basis is a set of eigenvectors for every power T^n.$\blacksquare$

This can also be shown without an appeal to the Spectral Theorem on diagonalizing normal operators. The definition of a normal operator implies that T and T^* commute, and therefore any monomial which has p factors of T and q factors of T^* is equal to $T^p(T^*)^q$. We also know that $(T^n)^* = (T^*)^n$, and if we combine these we see that

$$(T^n)^*T^n \quad = \quad (T^*)^nT^n \quad = \quad T^n(T^*)^n \quad = \quad T^n(T^n)^*$$

and therefore T^n is normal.$\blacksquare$