SOLUTIONS TO AXLER, SECTION 5A

2. Need to show that if $Sv = 0$ then $STv = 0$.
$Sv = 0 \Rightarrow TSv = 0 \Rightarrow STv = TSv = 0$. ∎

3. Need to show that $b = Sa \Rightarrow Tb = Sa'$, some a'.
But $Tb = TSa = STa$, so we can take $a' = Ta$. ∎

4. $T[U_i] \subseteq U_i$ all i. If $x \in \sum U_i$ then $x = \sum x_i$ where $x_i \in U_i$, so $Tx = \sum Tx_i \in \sum U_i$. ∎

6. Suppose $U \neq 0$ and $T[U] \subseteq U$ for every T. Let $0 \neq u_1 \in U$ and expand to a basis w/ $u_2, \ldots, u_n$.
Say $T_i : V \to V$ satisfies $T(u_1) = u_i$; we don't care what it does to other basis vectors. Then $u_i \in U$ for all $i \Rightarrow$ the basis $\{u_1, \ldots, u_n\}$ is in $U \Rightarrow U = V$. ∎

9. Matrix for T is $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Hence the eigenvalues are 0 and 5. The eigenspace for 0 is the kernel, which is generated by the unit column vector $E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. For 5, the eigenspace is $E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. ∎

11. <u>Translation</u>: For which polynomials $p(t)$ and scalars λ is $p'(t) = \lambda p(t)$?

Since $\deg p' < \deg p$ if p is a poly of pos. degree, we know p must be a constant polynomial ($\deg p' = \deg p - 1$, so $p' \neq \lambda p$ for $\lambda \neq 0$).
Every constant polynomial satisfies $(c)' = 0 = 0c$, so the only eigenvectors are constants & the only eigenvalue is 0. ∎

15. Do both at once. $Tv = cv$ & $v \neq 0$ $\Rightarrow$ $S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}cv = cS^{-1}v$. S^{-1} invertible $\Rightarrow$ $S^{-1}v \neq 0$, so $S^{-1}v$ is an eigenvector for eigenvalue c.
<u>CONVERSELY</u>, say $S^{-1}TSw = cw$. Apply S to both sides. This yields $TSw = Scw = cSw$, so Sw is an eigenvector of T with eigenvalue c. ∎

18. Suppose $Tz = cz$. Since $T(z_1, \dots) = (cz_1, \dots) = (0, z_1, \dots)$ we must have $c = 0$. However if $z \neq 0$ then $Tz \neq 0$, so 0 also can't be an eigenvalue. ∎

21. First of all if T (and T^{-1}) is invertible then $\operatorname{Ker} T = \operatorname{Ker} T^{-1} = \{0\}$ so 0 is an eigenvalue of neither.

(a) If λ is an eigenvalue for T and $Tv = \lambda v$ $(v \neq 0)$ then $v = T^{-1}\lambda v \Rightarrow \lambda^{-1} v = T^{-1} v$. Conversely, $\lambda^{-1} v = T^{-1} v \Rightarrow v = T^{-1}\lambda v \Rightarrow Tv = \lambda v$. ∎

(b) The preceding shows that v is a T-eigenvector with eigenvalue λ, then v is a T^{-1}-eigenvector with eigenvalue λ^{-1}. ∎

SOLUTIONS TO AXLER, SECTION 5B

2. Suppose $Tv = \lambda v$. Then

$$(T-2I)(T-3I)(T-4I)v =$$

$$(T-2I)(T-3I)(Tv-4v) = (T-2I)(T-3I)[(\lambda-4)v] =$$

$$(T-2I)[\lambda(\lambda-4)v - 3(\lambda-4)v] = (T-2I)[(\lambda-3)(\lambda-4)v] =$$

$$\lambda(\lambda-3)(\lambda-4)v - 2(\lambda-3)(\lambda-4)v =$$

$(\lambda-2)(\lambda-3)(\lambda-4)v$. If $(T-2I)(T-3I)(T-4I) = 0$, then this vector is 0; since $v \neq 0$ this means $(\lambda-2)(\lambda-3)(\lambda-4) = 0$, so that $\lambda = 2, 3$ or 4. ∎

6. First prove for powers of T. Go by induction T^d: $d = 1$, we are given $T[U] \subseteq U$.

If $T^{d-1}[U] \subseteq U$ then $T^d[U] = T^{d-1}[TU] \subseteq T^{d-1}[U] \subseteq U$. Now $I[U] = U$ and $S[U] \subseteq U$ $\Rightarrow aS[U] \subseteq U$, so if $p(T) = \sum a_i T^i$ we have $p(T)[U] \subseteq \sum a_i T^i[U] \subseteq \sum T^i[U] \subseteq U$. ∎

9. If 3 or -3 is an eigenvalue of T and $Tv = \pm 3v$ then $T^2_v = (\pm 3)^2 v = 9v$.

$v \neq 0$

Conversely, if $T^2 v = 9v$, consider $(T^2 - 9I)v = (T+3I)(T-3I)v = 0$. If v is not an ~~eigenvalue~~ eigenvector for 3 then $(T-3I)v \neq 0$ and if w is this vector, then $(T+3I)w = 0$, so -3 is an eigenvalue with associated eigenvector $w \neq 0$. ∎

14. If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then the diagonal entries are zero and $A^2 = I$, so $A = A^{-1}$. ∎

15. The zero matrix is the simplest example! ∎

20. Put T into triangular form w.r.t. ordered basis $w_1, w_2, \cancel{w_3}, \ldots w_n$. ~~Say the matrix is~~

$$\cancel{\begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix} \Rightarrow \text{Span } w_1 \text{ is a 1-dim}} \text{ inv't subspace.}$$

Let $V_i = \operatorname{span}\{w_1, \ldots w_i\}$. ∎

Solutions to Axler Section 5C

1. Let $v_1, \dots, v_p$ be a basis for the eigenspace of 0 = nullspace, and let $w_1, \dots, w_q$ be the remaining eigenvector basis s.t. all eigenvalues will be nonzero. Then the vector $Tw_j = c_j w_j$ form a basis for the image of T, so the vectors w_j do too. Since the v's form a basis for null T and the w's for range T, we have

$$V \cong \text{null}\, T \oplus \text{range}\, T.$$

2. Counterexample $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ Kernel = Image = span E_1.

3. Use rank + nullity = dim V.

(a) $\Rightarrow$ (b) automatic
(b) $\Rightarrow$ (c) $\dim V = \text{rank}\, T + \text{null}\, T \Rightarrow V = \text{Ker} + \text{Im}$

then $\dim V = \dim \text{Ker} + \text{Im} = \text{rank} + \text{nullity}$

$= \dim \text{Image} + \dim \text{Ker}$. On the other hand

$\dim V = \dim \text{Ker} + \text{Im} = \dim \text{Ker} + \dim \text{Im} -$

$\dim \text{Ker} \cap \text{Im} \Rightarrow 0 = \dim \text{Ker} \cap \dim \text{Im} \Rightarrow$

$\text{Ker} \cap \text{Im} = \{0\}$.

(c) $\Rightarrow$ (a) Need only show $V = \operatorname{Im} T + \operatorname{Ker} T$.

$\dim(\operatorname{Im} T + \operatorname{Ker} T) = \operatorname{rank} T + \operatorname{nullity} T - \dim(\operatorname{Im} T \cap \operatorname{Ker} T) = \operatorname{rank} T + \operatorname{nullity} T = \dim V.$

So $\dim(\operatorname{Im} T + \operatorname{Ker} T) = \dim V$, which means that $\operatorname{Im} T + \operatorname{Ker} T$ is all of V. $\blacksquare$

8. Given that dim (eigenspace for 4) = 4, $\dim V = 5$.

If neither is invertible then there are eigenvectors for 2 & 6, say x & y, in addition to the 4 basis vectors for the eigenspace of 8, say $z_1, \ldots, z_4$. Now x is not a lin comb of z_1, z_2, z_3, z_4 $\Rightarrow$ y is a lin comb of $x, z_1, \ldots, z_4$. But then ~~$Ty = 2x$~~ $y = ax + \sum b_i z_i$ $\Rightarrow$

$Ty = 2ax + \sum 8b_i z_i \neq 6y$, contradiction. $\blacksquare$
Hence either 2 or 6 is not an eigenvalue. $\blacksquare$

9. This is just Exercise 5A.21 in equivalent language. $\blacksquare$

12. We have bases $\{x_1, x_2, x_3\}$ & $\{y_1, y_2, y_3\}$ such that $Rx_1 = 2x_1$, $Rx_2 = 6x_2$, $Rx_3 = 7x_3$ and $Ty_1 = 2y_1$, $Ty_2 = 6y_2$, $Ty_3 = 7y_3$

Let S be the invertible linear transformation sending x_i to y_i & let $c_1, c_2, c_3 = 2, 6, 7$.

Then $S^{-1}TSx_i = S^{-1}Ty_i = S^{-1}c_iy_i = c_iS^{-1}y_i = c_ix_i = Rx_i$. Hence R and $S^{-1}TS$ agree on a basis, and hence they agree everywhere. ∎

16. (a) The matrix for T is $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

So $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ and if $T^n\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ then $T^{n+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix}$. ∎

(b) Use determinants: $\begin{vmatrix} -t & 1 \\ 1 & 1-t \end{vmatrix} = t^2 - t - 1 = 0$

roots are $\frac{1 \pm \sqrt{5}}{2}$.

(c) Find the nullspaces of

$$\begin{pmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$$ for the given values of λ.

This is messy but elementary, and one gets the following:

$$\lambda_+ = \frac{1+\sqrt{5}}{2} \quad \underbrace{\begin{pmatrix} 1 \\ \frac{1}{2}(1+\sqrt{5}) \end{pmatrix}}_{= v} \qquad \lambda_- = \frac{1-\sqrt{5}}{2} \quad \underbrace{\begin{pmatrix} 1 \\ \frac{1}{2}(1-\sqrt{5}) \end{pmatrix}}_{= w}$$

(d) Idea: Write $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as a linear comb. of v and w, say $v = av + bw$. Then

$$T^n v = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \lambda_+^n a v + \lambda_-^n b w.$$

The first coord of the expression on the right will be the formula for F_n.

(e) For this and details on d see

order-5. screenshot