(d) Denote $\left(1, \frac{1+\sqrt{5}}{2}\right)$ and $\left(1, \frac{1-\sqrt{5}}{2}\right)$ by e_1 and e_2 respectively. Then we have

$$(0,1) = \frac{1}{\sqrt{5}}(e_1 - e_2).$$

It follows that

$$\begin{aligned} T^n(0,1) =& T^n\left(\frac{1}{\sqrt{5}}(e_1 - e_2)\right) = \frac{1}{\sqrt{5}}(T^n e_1 - T^n e_2) \\ =& \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n e_1 - \left(\frac{1-\sqrt{5}}{2}\right)^n e_2\right]. \end{aligned}$$

By (a) and comparing the first component, we deduce that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{8}$$

(e) Note that $\sqrt{5} \geqslant 2$, we have

$$\frac{1}{\sqrt{5}}\left|\frac{1-\sqrt{5}}{2}\right|^n = \frac{1}{\sqrt{5}}\left|\frac{2}{1+\sqrt{5}}\right|^n \leq \frac{1}{2} \times \frac{2}{3} < \frac{1}{2}. \tag{9}$$

Moreover, $F_n \in \mathbb{Z}$ is easily shown by induction. Combining (8) and (9), it follows that

$$\left|\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - F_n\right| = \frac{1}{\sqrt{5}}\left|\frac{1-\sqrt{5}}{2}\right|^n < \frac{1}{2}$$

By $F_n \in \mathbb{Z}$, we deduce that the Fibonacci number F_n is the integer that is closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$