SOLUTIONS TO AXLER, SECTION 6A

2. Check that if $e_2 = (0,1,0)$, then $\langle e_2, e_2\rangle = 0$. ∎

4. (a) $\langle u+v, u-v\rangle = |u|^2 + \underbrace{\langle v,u\rangle - \langle u,v\rangle}_{\text{These cancel.}} + |v|^2$

$= |u|^2 - |v|^2$. ∎

8. Enough to show $u - v = 0$. But

$$|u-v|^2 = |u|^2 - 2\langle u,v\rangle + |v|^2 = 1 - 2 + 1 = 0.$$

Hence $u - v = 0$. Note this also works over $\mathbb{C}$, for $\langle u,v\rangle = 1 \Rightarrow \langle v,u\rangle = 1$. ∎

10. Let $w_1 = (1,3)$ and $w_2 = (3,-1)$. Then $w_1 \perp w_2$ and we have

$$(1,2) = v = \frac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle} w_1 + \frac{\langle v, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

Substitute into this formula, obtaining

$$(1,2) = \frac{7}{10}(1,3) + \frac{1}{10}(3,-1).$$

You should check that the RHSide is $(1,2)$. ∎

11. Apply the Schwarz $\leq$ to the vectors

$$(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}) \text{ and } \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}\right).$$

Note that their inner product is 4 and the lengths are the sqrts of the two expressions on the RH Side. Square the resulting $\leq$ to get the conclusion. $\boxtimes$

12. Same idea, but apply to $(1, \ldots, 1)$ and $(x_1, \ldots, x_n)$. $\boxtimes$

16. Use the ~~P~~ Parallelogram Law:

$$2\left(|u|^2 + |v|^2\right) = |u-v|^2 + |u-v|^2.$$

Given $|u+v| = 4$, $|u-v| = 6$, $|u| = 3$, we get

$$2(9 + |v|^2) = 16 + 36 = 52$$
$$9 + |v|^2 = 26$$
$$|v|^2 = 17, \text{ so } |v| = \sqrt{17}. \ \boxtimes$$

20. Add up the terms in the numerator using the rules

$$|x+y|^2 = |x|^2 + 2\operatorname{Re}\langle x, y\rangle + |y|^2$$
$$\operatorname{Re} i(a+ib) = -\operatorname{Im}(a+ib)$$
$$\operatorname{Im} i(a+ib) = \operatorname{Re}(a+ib).$$

After computing, you will see the sum is $4\langle x, y\rangle$. $\boxtimes$

23. The calculations can be simplified by means of an inductive argument on m. If $m=1$ the result is trivial. If the result is true for $m-1 \geq 1$, then $V_1 \times \dots \times V_m \cong W \times V_m$ where $W = V_1 \times \dots \times V_{m-1}$ and we know the result for the inner product on W by the induction hypothesis. Now verify the defining conditions for an inner product on $W \times V_m$. ∎

27. See axler6A.screenshot.png

31. Follow the drawing in axler6A.screenshot2.png.

The parallelogram law implies that

$$\underset{a^2}{2|u|^2} + \underset{b^2}{2|v|^2} = \underset{c^2}{|u-v|^2} + \underset{(2d)^2}{|u+v|^2}$$

since $d = \left|\frac{1}{2}(u+v)\right|$

Hence $2a^2 + 2b^2 = c^2 + 4d^2$ or

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2. \blacksquare$$

SOLUTIONS TO AXLER, SECTION 6B

1 (a) In each case the length of the vector is $\sqrt{\cos^2\theta+\sin^2\theta}=1$. For the first pair, the inner product of the two vectors is $-\cos\theta\sin\theta+\sin\theta\cos\theta=0$ and for the second it is $\cos\theta\sin\theta-\sin\theta\cos\theta=0$. ■

(b) Suppose we are given (a,b) & (c,d) so that $a^2+b^2=1=c^2+d^2$. Then $-1\le a,b,c,d\le 1$.

In particular, if $\theta_0=\arccos a$, then disregard
$a=\cos\theta_0$ and $b=\pm\sqrt{1-\cos^2\theta_0}=\pm\sin\theta_0$.

If the sign is $+$, take $\theta=\theta_0$ to get $b=\sin\theta$,

and if the sign is $-$, take $\theta_0=-\theta$ to get

$a=\cos\theta\ (=\cos-\theta_0)$, $b=\sin\theta=-\sin\theta_0$.

We now have $c\cos\theta+d\sin\theta=0$, where $c^2+d^2=1$.

The solutions for this equation are all (c,d) which are multiples of $(-\sin\theta,\cos\theta)$, so

$(c,d)=(-K\sin\theta,K\cos\theta)$ for some constant K.

But now $c^2+d^2=1=K^2(\sin^2\theta+\cos^2\theta)=K^2$

$\Rightarrow K=\pm1$. So $(c,d)=(\mp\sin\theta,\pm\cos\theta)$. ■

2. Suppose $v = \sum c_j e_j$. Then as before we have $\langle v, e_i \rangle = c_i$ (see math 132notes 6Ba).
and in fact $|v|^2 = \sum |c_j|^2 = \sum |\langle v, e_j\rangle|^2$.

Now suppose v is not a lin comb. as above. Extend $e_1, \ldots, e_m$ to an orthonormal basis, say $e_1, \ldots, e_m, e_{m+1}, \ldots e_n$. Once again we have $|v|^2 = \sum |\langle v, e_j\rangle|^2$ but now some $c_k \neq 0$ ~~since~~ for some $k > m$ and hence

$$|v|^2 = \sum_1^n |\langle v, e_j\rangle|^2 > \sum_1^m |\langle v, e_j\rangle|^2.$$

Thus we have HYPOTHES TRUE $\Rightarrow$

CONCLUSION TRUE and HYPOTHESIS FALSE $\Rightarrow$ CONCLUSION FALSE, or equivalently HYPOTHESIS TRUE $\Leftrightarrow$ CONCLUSION TRUE. $\boxtimes$

3. By 6.37 and 6.38 it suffices to convert the given basis into an orthonormal basis by the Gram-Schmidt algorithm.

To simplify computations, we first find an orthogonal set by Gram-Schmidt.

$w_1 = (1, 0, 0)$ $\quad w_2 = (1,1,1) - (1, 0,0) =$
$(0,1,1)$

$w_3 = (1,1,2) - \frac{1}{1}(1,0,0) - \frac{3}{2}(0,1,1) =$
$(0, -\frac{1}{2}, \frac{1}{2})$

Now divide these by their lengths to get the associated unit vectors.

$u_1 = (1,0,0)$ $\quad u_2 = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$u_3 = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. $\blacksquare$

10. At each step there are two choices for w such that $\{v_1, \ldots, v_k, v_{k+1}\}$ spans the same subspace as $\{e_1, \ldots, e_k, e_{k+1}\}$ provided $e_1, \ldots, e_k$ are held fixed. Specifically, $w = \pm e_{k+1}$.

This is true because we have dim span $\{e_1, \ldots, e_k\} = k$, dim span $\{v_1, \ldots, v_{k+1}\} = k+1$, so the orthogonal complement of the first subspace in the second is 1-dimensional.

Since there are 2 choices at each step, there are a total of 2^m choices for $\{e_1, \ldots, e_m\}$. $\blacksquare$

SOLUTIONS TO AXLER, SECTION 6C

3. Everything in Gram-Schmidt implies that $e_1, \dots, e_m$ is an orthonormal basis for U, and each of $f_1, \dots, f_n$ lies in $U^\perp$. Since $\dim U^\perp = \dim V - \dim U = (m+n) - m = n$, the set $\{f_1, \dots, f_n\}$, which by construction is linearly indep., must be a basis for $U^\perp$. $\boxed{}$

4. Use the preceding exercise with $u_1 = (1, 2, 3, -4)$ and $u_2 = (-5, 4, 3, 2)$
Let $w_1 = (0, 0, 1, 0)$ and $w_2 = (0, 0, 0, 1)$. One can then show $\{u_1, u_2, w_1, w_2\}$ is lin indep (hence a basis) by row reducing

$$\begin{pmatrix} 1 & 2 & 3 & -4 \\ -5 & 4 & 3 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to the identity (details omitted).
We shall only find an orthogonal basis; divide by vector length to get an orthonormal one.

x_1, x_2 basis for U

y_1, y_2 basis for $U^\perp$

5. $E = \perp$ proj onto U. Show $I - E = \perp$ proj onto $U^\perp$.

Write $v = v_1 + v_2$ where $v_1 \in U$, $v_2 \in U^\perp$. By assumption, $Ev = v_1$. But this means

$$(I - E)(v_1 + v_2) = v_1 + v_2 - v_1 = v_2. \blacksquare$$

10.
11. } see axler6C.pdf with annotations in red to clarify the discussion.