Suppose $u \in U$. Then $P_U u = u$, so applying both sides of the equation above to u gives $P_U(Tu) = Tu$, which implies that $Tu \in U$. Because u was
Disregard ^ an arbitrary vector in U, this implies that T is invariant under U, as desired.

10 ~~20~~ Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U and $U^\perp$ are both invariant under T if and only if $P_U T = T P_U$.

SOLUTION: First suppose that U and $U^\perp$ are both invariant under T. By the previous exercise, this implies that

$$P_U T P_U = T P_U$$

and

$$P_{U^\perp} T P_{U^\perp} = T P_{U^\perp}.$$

But $P_{U^\perp} = I - P_U$, so the last equation becomes

$$(I - P_U)T(I - P_U) = T(I - P_U).$$

Expanding both sides of the equation above and rearranging terms, we get

$$P_U T P_U = P_U T.$$

Combining this with the first equation above, we get $P_U T = T P_U$, as desired.

To prove the implication in the other direction, suppose now that

$$P_U T = T P_U.$$

Then

$$\begin{aligned} P_U T P_U &= (P_U T) P_U \\ &= (T P_U) P_U \\ &= T {P_U}^2 \\ &= T P_U, \end{aligned}$$

which implies (by the previous exercise) that U is invariant under T, as desired. Also,

$$\begin{aligned} P_{U^\perp} T P_{U^\perp} &= \big((I - P_U)T\big) P_{U^\perp} \\ &= (T - P_U T) P_{U^\perp} \\ &= (T - T P_U) P_{U^\perp} \\ &= T(1 - P_U) P_{U^\perp} \\ &= T {P_{U^\perp}}^2 \\ &= T P_{U^\perp}, \end{aligned}$$

which implies (by the previous exercise) that $U^\perp$ is invariant under T, as desired.

11 ~~21~~ In $\mathbf{R}^4$, let

$$U = \operatorname{span}\big((1,1,0,0),(1,1,1,2)\big).$$

Find $u \in U$ such that $\|u - (1,2,3,4)\|$ is as small as possible.

SOLUTION: First we find an orthonormal basis of U by applying the Gram-Schmidt procedure to $\big((1,1,0,0),(1,1,1,2)\big)$, getting

$$e_1 = \Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\Big)$$

$$e_2 = \Big(0, 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\Big).$$

Thus with e_1, e_2 as above, (e_1, e_2) is an orthonormal basis of U. By 6.36 and 6.35, the closest point $u \in U$ to $(1,2,3,4)$ is

$$\langle (1,2,3,4), e_1\rangle e_1 + \langle (1,2,3,4), e_2\rangle e_2,$$

which equals

$$\Big(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5}\Big).$$

22. Find $p \in \mathcal{P}_3(\mathbf{R})$ such that $p(0) = 0$, $p'(0) = 0$, and

Disregard all this.

$$\int_0^1 |2 + 3x - p(x)|^2\,dx$$

is as small as possible.

SOLUTION: Define an inner product on $\mathcal{P}_3(\mathbf{R})$ by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Let $q(x) = 2 + 3x$, and let

$$U = \{p \in \mathcal{P}_3(\mathbf{R}) : p(0) = 0, p'(0) = 0\}.$$

With this notation, our problem is to find the closest point $p \in U$ to q. To do this, first we find an orthonormal basis of U.