SOLUTIONS TO AXLER, SECTION 7A

1. The linear transformation's behavior on standard unit vectors is easy to describe.
$T(e_1) = e_2,\ T(e_2) = e_3, \ldots, Te_{n-1} = e_n, Te_n = 0.$
Hence the associated matrix $A = (a_{ij})$ has entries $\langle Te_i, e_j \rangle = \begin{matrix} 1 & j = i+1,\ 1 \le j \le n \\ 0 & \text{otherwise.} \end{matrix}$
The entries of the associated matrix A^* for T^* will then have entries b_{ij} where

$$b_{ij} = a_{ji} = \begin{cases} 1 & i = j+1 \\ 0 & \text{otherwise} \end{cases}$$

So
$$A = \begin{pmatrix} 0 & 0 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & \vdots \\ \vdots & \vdots & & 1 & 0 \\ 0 & & & & \end{pmatrix}$$ and

$$A^* = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & & 0 \end{pmatrix} = B. \blacksquare$$

3. Suppose $T[U] \subseteq U$. Then $u \in U \Rightarrow \langle u, x\rangle = 0$ all $x \in U^\perp$ and $Tu \in U \Rightarrow \langle Tu, x\rangle = 0$ all such x. But now $\langle Tu, x\rangle = \langle u, T^*x\rangle \Rightarrow$ if $x \in U^\perp$, then $T^*x \in U^\perp$.
The converse also follows, for $T^*[U^\perp] \subseteq U$ and the previous argument imply $T^{**}[U^{\perp\perp}] \subseteq U^{\perp\perp}$. Since $T^{**} = T$ and $U^{\perp\perp} = U$ we have that $T^*[U^\perp] \subseteq U^\perp \Rightarrow T[U] \subseteq V$. $\blacksquare$

4. Note that (a) $\Rightarrow$ (b) because $T = T^{**}$.
So we are reduced to proving (a):
If $\operatorname{Ker} T = 0$ then $0 = \operatorname{Ker} T = (\operatorname{Im} T^*)^\perp$ by Axler, p. 207. Since $W^\perp = 0 \Leftrightarrow W = V$, this means $\operatorname{Im} T = V$. $\blacksquare$ [U subspace V]

7. Suppose $ST = TS$. Then $(ST)^* = T^*S^* = TS$ (self adjointness) $= ST$, so ST is also self adjoint. Conversely, suppose $ST = TS$.
~~and $(ST)^* = ST$. Then $(ST)^* = ST = T^*$~~
Then $(ST)^* = T^*S^* = TS = ST$, so ST is also self-adjoint.

12. Given 3 + 4 are eigen~~vectors~~values & T has an orthonormal basis of eigenvectors Then there must be some orthonormal basis s.t. $Tu_1 = 3u_1$, $Tu_2 = 4u_2$. Then $T(u_1+u_2) = 3u_1 + 4u_2$, so $|T(u_1+u_2)| = 5$. Also $|u_1+u_2| = \sqrt{2}$. (just use the Pythagorean formula to show $x_1u_1 + x_2u_2 = \sqrt{x_1^2 + x_2^2}$. $\square$

SOLUTIONS TO AXLER, SECTION 7B

2. Since T has a basis of eigenvectors we have a basis $u_1, \dots, u_p, v_1, \dots, v_q$ so that $Tu_i = 2u_i$ & $Tv_j = 3v_j$ all i & j. Given $x \in V$, write $x = \sum a_i u_i + \sum b_j v_j$.
Let $p(t) = t^2 - 5t + 6 = (t-2)(t-3)$
Then $p(T)x = \sum p(2) a_i u_i + \sum p(3) b_j v_j$. This is zero because $p(2) = p(3) = 0$. ∎

3. Take T with standard matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

Then the eigenvectors for 2 are scalar mults. of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and for 3 they are scalar multiples of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
We need to calculate $p(A)$.

(see next page)

To see it's nonzero we can check what it does on a basis, say the unit vectors E_1, E_2, E_3. We have $AE_1 = 2E_1$, $AE_2 = 3E_2$, $AE_3 = 3E_3 + E_2$. We then have

$$p(A)E_1 = p(2)E_1 = 0 \cdot E_1 = 0.$$

$$p(A)E_2 = p(3)E_2 = 0 \cdot E_2 = 0.$$

$$p(A)E_3 = \overbrace{A(3E_3 + E_2)}^{A^2E_3} - \overbrace{5(3E_3+E_2)}^{5AE_3} + 6E_3 =$$

$$\underbrace{3(3E_3+E_2) + 3E_2}_{A^2E_3} - 15E_3 - 5E_2 + 6E_3 =$$

$$\underbrace{(9 - 15 + 6)}_{0}E_3 + (3+3-5)E_2 = 4E_2 \neq 0.$$

Therefore $p(A) \neq 0$ and hence the same is true for $p(T)$.

NOTE If we let $q(t) = (t-3)^2(t-2)$, then we have $q(T) = 0$ (see what happens on E_3 in this case!).

One can also see this by computing $p(A)$ directly. ▣

6. We know that a normal operator has an orthonormal basis of eigenvectors $u_1, \ldots, u_n$ with eigenvalues $c_1, \ldots, c_n$. If $T^* = T$ the c_j are all real, and conversely, if T is normal and the eigenvalues are real we have (if $x = \sum a_j u_j$) $T^* x = \sum a_j T^* u_j =$

$$\sum a_j \bar{c}_j \cdot u_j \underset{\text{since } c_j \text{ real}}{=\!=\!=} \sum a_j c_j u_j = Tx, \text{ so}$$

$T = T^*$ since x was arbitrary. ▣

15. Write $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & x \end{pmatrix}$ where we want to find x from

$A \cdot A^T = A^T \cdot A$. Strictly speaking we should find all 9 entries of each product matrix and compare them. In the (1,3) entry we get the condition $1 \; \& \; x$, and A normal $\Rightarrow$ these equal $\Rightarrow x = 1$. ▣

(see the file axler7B15.png for more details)

SOLUTIONS TO AXLER, SECTION 7C

2. We shall show $v - w = 0$.
Since T is positive $\langle T(v-w), v-w \rangle \geq 0$
~~Left hand side is~~ Also $Tv = w$, $Tw = v \Rightarrow$
$T(v-w) = w - v = -(v-w)$. Hence
$\langle T(v-w), v-w \rangle = \langle -(v-w), (v-w) \rangle =$
$-|v-w|^2$, which is ≤ 0. The only possibility is that $|v-w| = 0$, so that $v = w$. $\blacksquare$

1. FALSE. Take $\dim V = 2$, $\{e_1, e_2\}$ orthonormal basis, T with matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
Then $\langle T(e_1 - e_2), (e_1 - e_2) \rangle \underset{\text{since } T \text{ selfadjoint}}{=}$
$\langle Te_1, e_1 \rangle - 2 \langle Te_1, e_2 \rangle + \langle Te_2, e_2 \rangle$
$1 - 2 \cdot 2 + 1 = -2$. $\blacksquare$

5. $S+T$ positive $\Rightarrow S, T$ self adjoint $\Rightarrow$ so is $S+T$.

Since $\langle Sv, v\rangle, \langle Tv, v\rangle \geq 0$ all v we have

① $\langle [S+T]v, v\rangle = \langle Sv + Tv, v\rangle =$

$\langle Sv, v\rangle + \langle Tv, v\rangle$ which is nonneg since both summands are. Furthermore, this sum is zero $\Leftrightarrow v=0$ because $\langle Sv, v\rangle = 0 = \langle Tv, v\rangle$ by positivity. Hence $S+T$ is also positive.

6. If k is even, then $k=2m \Rightarrow$

$\langle T^k v, v\rangle = \langle T^m v, T^m v\rangle$ since T is self-adjoint. But RH Side is positive definite, so this is $0 \Leftrightarrow T^m v = 0$. Since T is invertible, we must have T^m invertible and hence T^k is positive definite.

Now suppose $k = 2m+1$, or k is odd. Then

$$\langle T^k v, v\rangle = \langle T(T^m v), T^m v\rangle$$

which is nonnegative and $0 \iff T^m v = 0$. Now again T pos definite $\Rightarrow T$ invertible $\Rightarrow$ every $x \in V$ is $T^m v$ for some unique v. Hence $T^m v = 0 \iff v = 0$. Combining these T^k is selfadjoint, and $\langle T^k v, v\rangle \geq 0$ with equality $\iff v = 0$. ∎

9. TRUE. Take reflection in $\mathbb{R}^2$ about the line joining $(0,0)$ to $(\cos\theta, \sin\theta)$, which is given by $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$. Standard trigonometric identities imply that the square of this matrix is I. Different choices of $\theta \in [0, \pi)$ yield different matrices, so this is an infinite family of 2×2 matrices whose squares $= I$. — Extend to larger matrices by taking block sums with an $(n-2)\times(n-2)$ identity matrix. ∎

Block sum: $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ A is $p\times p$, B is $q\times q$