SOLUTIONS TO EXERCISES IN AXLER, SECTION 8A.

1. Check that $T^2 = 0$, so that every nonzero vector is a generalized eigenvector with eigenvalue 0. ■

5. Suppose $v, \dots, T^{m-1}v$ is linearly dependent, and write $\sum_{0 \le j \le m-1} a_j T^j v = 0$ where not all a_j's are 0. Apply T^{m-1}, so $a_0 T^{m-1} v = 0$; since $T^{m-1} v \neq 0$, we have $a_0 = 0$. Similarly
$$0 = T^{m-2} 0 = \underbrace{\cancel{a_0 T^{m-2} v}}_{0} + a_1 T^{m-1} v + \underbrace{\cancel{T^{m}(\text{junk})}}_{0} = 0$$
so $a_1 T^{m-1} v = 0$ and $a_1 = 0$. Continue in this manner to show $0 = a_2 = \dots = a_{m-1}$. CONTRADICTION. Therefore $\{v, \dots, T^{m-1} v\}$ is lin indep. ■

8. $S(u,v) = (v, 0)$, $T(u,v) = (0, u) \Rightarrow S^2 = T^2 = 0$
However $S + T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is not nilpotent. ■
$(S+T)^2 = I$.

9. Suppose $(ST)^n = 0$. Then
$$(TS)^{n+1} = T(ST)^n S = 0. \ ■$$

13. N normal + nilpotent. $N^p = 0 \Rightarrow 0$ is the only eigenvalue. $NN^* = N^*N \Rightarrow N$ has a basis of eigenvectors, say v_i. Combining these, $Nv_j = 0$ all j, which means that $Nv = 0$ all v. ■

SOLUTIONS TO EXERCISES IN AXLER SECTION 8B

1. If N is the only eigenvalue and N is triangular form, then $N^p = 0$. For a general N we know $N = P^{-1}AP$ where A is upper triangular; since A also has only 0 as an eigenvalue, we have $A^p = 0$ (some p) hence

$$N^p = (P^{-1}AP)^p = P^{-1}A^pP = P^{-1}0P = 0. \blacksquare$$

3. Suppose that $x_1, \ldots, x_k$ is a basis for the eigenvectors of T with eigenvalue c. Then $W = \text{span}$ $\Longrightarrow$ $S^{-1}[W]$ & W have the same dimension, and in fact $S^{-1}x_1, \ldots, S^{-1}x_k$ satisfy

$$S^{-1}TS(S^{-1}x_j) = S^{-1}TS(cS^{-1}x_j) = cS^{-1}x_j.$$

Hence if W is as above then $S^{-1}[W] \subseteq$ space X of eigenvectors for $S^{-1}TS$ with eigenvalue c. Hence $\dim W \leq \dim X$. Similarly, if we consider $S(S^{-1}TS)S^{-1}$ we see $\dim X \leq \dim W$. Thus $\dim X = \dim W$. $\blacksquare$

6. In matrix form,

$$N = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that $N^5 = 0$ and ~~solve~~ find "$\sqrt{I+N}$" using the Binomial Series Formula. This converges because all higher powers are zero.

$$"\sqrt{I+N}" = I + \frac{1}{2}N + \frac{1(-1)}{2\cdot 2\, 2}N^2 + \frac{1(-1)(-3)}{2\cdot 2\cdot 2\, 3!}N^3 +$$

$$\frac{1(-1)(-3)(-5)}{2\cdot 2\cdot 2\cdot 2\cdot 4!}N^4 =$$

$$I + \frac{1}{2}N - \frac{1}{8}N^2 + \frac{1}{16}N^3 - \frac{5}{128}N^4. \blacksquare$$

9. Write $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_m$ where n = size of A, B (# rows/columns), and size V_j = size A_j, B_j.

Then $A(y_1, \ldots, y_m) = (A_1 y_1, \ldots, A_m y_m)$ and

$$B(x_1, \ldots, x_m) = (B_1 x_1, \ldots, B_m x_m).$$

Combining these, we obtain

$$AB(x_1, \ldots, x_m) = A(B_1 x_1, \ldots, B_m x_m) =$$

$$(A_1 B_1 x_1, \ldots, A_m B_m x_m).$$ If $z \in V_j$ then view $z = (0, \ldots, \underset{\uparrow \text{ jth place}}{z}, \ldots 0)$ and notice that

$ABz = (0, \ldots, A_j B_j z, \ldots 0)$. Thus if we take an ordered basis $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_m$ for $\mathbb{R}^n$

it follows that the matrix of AB for this ordered basis is the block sum $A_1B_1 \oplus \cdots \oplus A_mB_m$. ⊠

11. If A is upper triangular and B is a Jordan form for A, then the assertions about the characteristic polynomial (not yet proven) imply $\chi_A = \chi_B$. Hence the numbers of factors $(t-\lambda)$ in both are the same (fixed λ). On the right hand side this number is the sum of the sizes of the Jordan blocks for λ. ⊠

This corrects the conclusion stated in Axler. The number of factors is the dimension of V_λ, NOT the dimension of the eigenspace for λ!

SOLUTIONS TO EXERCISES IN AXLER, SECTION 8C

→ characteristic poly has the

1. The minimal polynomial is divisible by $(t-3)(t-5)(t-8)$, and it has the ~~form~~

$$(t-3)^a(t-5)^b(t-8)^c \quad \text{where } a, b, c \geq 1$$

and $a+b+c=4$. This means that two of $\{a, b, c\}$ are 1 and one is 2. For all such possibilities we have $\chi \mid (t-3)^2(t-5)^2(t-8)^2$. Since $\chi(T)=0$ by the Cayley–Hamilton Thm., it follows that $(T-3I)^2(T-5I)^2(T-8I)^2=0$. $\blacksquare$

5. $$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \blacksquare$$

6. $$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \blacksquare$$

7. We know $m+n=\dim V$, $m=\dim V_0$. Hence $n=\dim V_1$, so $\chi(P)=t^m(t-1)^n$. $\blacksquare$

11. Let $p(t)$ be the minimal poly of T, $\chi(t)$ the char poly. for Jordan form. Then, over $\mathbb{C}$, $\chi(t)$ is a product $(t-\lambda_i)^{m_i}$ where $\lambda_i \neq 0$, so

~~ap~~ $p(t)$ is a poly $(t-\lambda_i)^{n_i}$ where $n_i \leq m_i$.

The constant term for $\chi(t)$ is nonzero $\Rightarrow$ same true for $p(T) = T^k + a_{k-1} T^{k-1} + \dots + a_0 I$.

It follows that $T^{-1} = \frac{1}{a_0} \left(\sum_{j=1}^{k} a_j T^{j-1} \right)$. ∎

SOLUTIONS TO EXERCISES FROM AXLER, SECTION 8D

The assignment simplifies these problems by adding the condition that $T: V \to V$ has a matrix which is an elementary Jordan matrix.

4. The assumption is that $Tv_k = \lambda v_k + v_{k-1}$ for $k \geq 2$ and $Tv_1 = \lambda v_1$. Reversing the order of the v's yields $w_j = v_{n-j+1}$, and $Tw_j = \lambda v_{n-j+1} + v_{n-j} = \lambda w_j + w_{j+1}$ if $n-j \geq 1$; i.e., if $j < n$. To complete the picture, $Tw_n = Tv_1 = \lambda v_1 = \lambda w_n$. ■

<u>Another description</u> The matrix A for T w.r.t. the first ordered basis satisfies

$$a_{ij} = \begin{cases} \lambda & i=j \\ 1 & j=i+1 \\ 0 & \text{otherwise.} \end{cases}$$

The matrix B w.r.t. the reordered basis is

$$b_{ij} = \begin{cases} \lambda & i=j \\ 1 & i=j+1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence B is the <u>transpose</u> of A. ■

5. $T = \lambda I + N$, where N is an elementary matrix in Jordan form.

$$N_{ij} = \begin{cases} 1 & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

3x3 example $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad (N^2)_{i,j} = \begin{cases} 1 & j = i+2 \\ 0 & \text{otherwise} \end{cases}$

example $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$T^2 = (\lambda I + N)^2 = \lambda^2 I + 2\lambda N + N^2 \Rightarrow$$

matrix looks like $\longrightarrow \begin{pmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}$

If C is this matrix then

$$c_{ij} = \begin{cases} \lambda^2 & i = j \\ 2\lambda & j = i+1 \\ 1 & j = i+2 \\ 0 & \text{otherwise} \end{cases} \quad \blacksquare$$