Solutions to problems from Axler, Section 10A

CHAPTER 10

Trace and Determinant

1. Suppose that $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V. Prove that $\mathcal{M}(T, (v_1, \ldots, v_n))$ is invertible if and only if T is invertible.

SOLUTION: First suppose that $\mathcal{M}(T)$ is an invertible matrix (because the only basis is sight is $(v_1, \ldots, v_n)$, we can leave the basis out of the notation). Thus there exists an n-by-n matrix B such that

$$\mathcal{M}(T)B = B\mathcal{M}(T) = I.$$

There exists an operator $S \in \mathcal{L}(V)$ such that $\mathcal{M}(S) = B$ (see 3.19). Thus the equation above becomes

$$\mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I,$$

which we can rewrite as

$$\mathcal{M}(TS) = \mathcal{M}(ST) = \mathcal{M}(I),$$

which implies that

$$TS = ST = I.$$

Thus T is invertible, as desired, with inverse S.

To prove the implication in the other direction, suppose now that T is invertible. Thus there exists $S \in \mathcal{L}(V)$ such that

$$TS = ST = I.$$

This implies that

$$\mathcal{M}(TS) = \mathcal{M}(ST) = \mathcal{M}(I),$$

which implies that

$$\mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I.$$

Thus $\mathcal{M}(T)$ is invertible, as desired, with inverse $\mathcal{M}(S)$.

2. Prove that if A and B are square matrices of the same size and $AB = I$, then $BA = I$.

SOLUTION: Suppose that A and B are n-by-n matrices and $AB = I$. There exist $S, T \in \mathcal{L}(\mathbf{F}^n)$ such that

$$\mathcal{M}(S) = A \quad \text{and} \quad \mathcal{M}(T) = B;$$

here we are using the standard basis of $\mathbf{F}^n$ (the existence of $S, T \in \mathcal{L}(\mathbf{F}^n)$ satisfying the equations above follows from 3.19). Because $AB = I$, we have $\mathcal{M}(S)\mathcal{M}(T) = I$, which implies that $\mathcal{M}(ST) = \mathcal{M}(I)$, which implies that $ST = I$, which implies that $TS = I$ (by Exercise 23 in Chapter 3). Thus

$$\begin{aligned} BA &= \mathcal{M}(T)\mathcal{M}(S) \\ &= \mathcal{M}(TS) \\ &= \mathcal{M}(I) \\ &= I. \end{aligned}$$

3. Suppose $T \in \mathcal{L}(V)$ has the same matrix with respect to every basis of V. Prove that T is a scalar multiple of the identity operator.

SOLUTION: We begin by proving that (v, Tv) is linearly dependent for every $v \in V$. To do this, fix $v \in V$, and suppose that (v, Tv) is linearly independent. Then (v, Tv) can be extended to a basis $(v, Tv, u_1, \ldots, u_n)$ of V. The first column of the matrix of T with respect to this basis is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Clearly $(2v, Tv, u_1, \ldots, u_n)$ is also a basis of V. The first column of the matrix of T with respect to this basis is

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus T has different matrices with respect to the two bases we have considered. This contradiction shows that (v, Tv) is linearly dependent for every $v \in V$. This implies that for every vector in V is an eigenvector of T. This implies that T is a scalar multiple of the identity operator (by Exercise 12 in Chapter 5).

4. Suppose that $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ are bases of V. Let $T \in \mathcal{L}(V)$ be the operator such that $Tv_k = u_k$ for $k = 1, \dots, n$. Prove that

$$\mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)).$$

SOLUTION: Fix k. Write

$$u_k = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \dots, a_n \in \mathbf{F}$. Because $Tv_k = u_k$, the k^{th} column of the matrix $\mathcal{M}(T, (v_1, \dots, v_n))$ consists of the numbers $a_1, \dots, a_n$. Because $Iu_k = u_k$, the k^{th} column of $\mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n))$ also consists of the numbers $a_1, \dots, a_n$.

Because $\mathcal{M}(T, (v_1, \dots, v_n))$ and $\mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n))$ have the same columns, these two matrices must be equal.

5. Prove that if B is a square matrix with complex entries, then there exists an invertible square matrix A with complex entries such that $A^{-1}BA$ is an upper-triangular matrix.

SOLUTION: Suppose B is an n-by-n matrix with complex entries. Let $(e_1, \dots, e_n)$ denote the standard basis of $\mathbf{C}^n$. There exists $T \in \mathcal{L}(\mathbf{C}^n)$ such that $\mathcal{M}(T, (e_1, \dots, e_n)) = B$ (see 3.19).

There is a basis $(v_1, \dots, v_n)$ of V such that $\mathcal{M}(T, (v_1, \dots, v_n))$ is an upper-triangular matrix (see 5.13). Let $A = \mathcal{M}((v_1, \dots, v_n), (e_1, \dots, e_n))$. Then A is invertible (by 10.2) and

$$\begin{aligned} A^{-1}BA &= A^{-1}\mathcal{M}(T, (e_1, \dots, e_n))A \\ &= \mathcal{M}(T, (v_1, \dots, v_n)), \end{aligned}$$

where the second equality comes from 10.3. Thus $A^{-1}BA$ is an upper-triangular matrix.

6. Give an example of a real vector space V and $T \in \mathcal{L}(V)$ such that

$$\operatorname{trace}(T^2) < 0.$$

SOLUTION: Define $T \in \mathcal{L}(\mathbf{R}^2)$ by

$$T(x, y) = (-y, x).$$

Then $T^2 = -I$, so $\operatorname{trace}(T^2) = -2$.

7. Suppose V is a real vector space, $T \in \mathcal{L}(V)$, and V has a basis consisting of eigenvectors of T. Prove that $\operatorname{trace}(T^2) \geq 0$.

SOLUTION: Let $(v_1, \dots, v_n)$ be a basis of V consisting of eigenvectors of T. Thus there exist $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ such that $Tv_j = \lambda_j v_j$ for each j. Clearly the matrix of T^2 with respect to the basis $(v_1, \dots, v_n)$ is the diagonal matrix

$$\begin{bmatrix} {\lambda_1}^2 & & 0 \\ & \ddots & \\ 0 & & {\lambda_n}^2 \end{bmatrix}.$$

Thus $\operatorname{trace} T^2 = {\lambda_1}^2 + \cdots + {\lambda_n}^2 \geq 0$.

8. Suppose V is an inner-product space and $v, w \in \mathcal{L}(V)$. Define $T \in \mathcal{L}(V)$ by $Tu = \langle u, v \rangle w$. Find a formula for $\operatorname{trace} T$.

SOLUTION: First suppose that $v \neq 0$. Extend $(\frac{v}{\|v\|})$ to an orthonormal basis $(\frac{v}{\|v\|}, e_1, \dots, e_n)$ of V. Note that for each j, we have $Te_j = 0$ (because $\langle e_j, v \rangle = 0$). The trace of T equals the sum of the diagonal entries in the matrix of T with respect to the basis $(\frac{v}{\|v\|}, e_1, \dots, e_n)$. Thus

$$\begin{aligned} \operatorname{trace} T &= \langle T(\frac{v}{\|v\|}), \frac{v}{\|v\|} \rangle + \langle Te_1, e_1 \rangle + \cdots + \langle Te_n, e_n \rangle \\ &= \langle \langle \frac{v}{\|v\|}, v \rangle w, \frac{v}{\|v\|} \rangle \\ &= \langle w, v \rangle. \end{aligned}$$

If $v = 0$, then $T = 0$ and so $\operatorname{trace} T = 0 = \langle w, v \rangle$. Thus we have the formula

$$\operatorname{trace} T = \langle w, v \rangle$$

regardless of whether or not $v = 0$.

9. Prove that if $P \in \mathcal{L}(V)$ satisfies $P^2 = P$, then $\operatorname{trace} P$ is a nonnegative integer.

SOLUTION: Suppose that $T \in \mathcal{L}(V)$ satisfies $P^2 = P$. Let $(u_1, \dots, u_m)$ be a basis of $\operatorname{range} P$ and let $(v_1, \dots, v_n)$ be a basis of $\operatorname{null} P$. Then

$$(u_1, \dots, u_m, v_1, \dots, v_n)$$

is a basis of V (this holds because $V = \operatorname{range} T \oplus \operatorname{null} T$; see Exercise 21 in Chapter 5). For each u_j we have $Pu_j = u_j$ and for each v_k we have $Pv_k = 0$. Thus the matrix of P with respect to the basis above of V is a diagonal matrix whose diagonal contains m 1's followed by n 0's. Thus $\operatorname{trace} P = m$, which is a nonnegative integer, as desired. In fact, we have shown that

$$\operatorname{trace} P = \dim \operatorname{range} P.$$

10. Prove that if V is an inner-product space and $T \in \mathcal{L}(V)$, then

$$\operatorname{trace} T^* = \overline{\operatorname{trace} T}.$$

SOLUTION: Suppose that V is an inner-product space and $T \in \mathcal{L}(V)$. Let $(e_1, \dots, e_n)$ be an orthonormal basis of V. The trace of any operator on V equals the sum of the diagonal entries on the matrix of the operator with respect to this basis. Thus

$$\begin{aligned}
\operatorname{trace} T^* &= \langle T^* e_1, e_1 \rangle + \cdots + \langle T^* e_n, e_n \rangle \\
&= \langle e_1, T e_1 \rangle + \cdots + \langle e_n, T e_n \rangle \\
&= \overline{\langle T e_1, e_1 \rangle} + \cdots + \overline{\langle T e_n, e_n \rangle} \\
&= \overline{\langle T e_1, e_1 \rangle + \cdots + \langle T e_n, e_n \rangle} \\
&= \overline{\operatorname{trace} T}.
\end{aligned}$$

11. Suppose V is an inner-product space. Prove that if $T \in \mathcal{L}(V)$ is a positive operator and $\operatorname{trace} T = 0$, then $T = 0$.

SOLUTION: Suppose $T \in \mathcal{L}(V)$ is a positive operator and $\operatorname{trace} T = 0$. There exists an operator $S \in \mathcal{L}(V)$ such that $T = S^* S$ (by 7.27). Let $(e_1, \dots, e_n)$ be an orthonormal basis of V. Then

$$\begin{aligned} 0 &= \operatorname{trace} T \\ &= \langle Te_1, e_1 \rangle + \cdots + \langle Te_n, e_n \rangle \\ &= \langle S^*Se_1, e_1 \rangle + \cdots + \langle S^*Se_n, e_n \rangle \\ &= \|Se_1\|^2 + \cdots + \|Se_n\|^2. \end{aligned}$$

The equation above implies that $Se_j = 0$ for each j. Because S is 0 on a basis of V, we have $S = 0$. Because $T = S^*S$, this implies that $T = 0$.

~~12.~~ **13.** Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is the operator whose matrix is

$$\begin{bmatrix} 51 & -12 & -21 \\ 60 & -40 & -28 \\ 57 & -68 & 1 \end{bmatrix}.$$

Someone tells you (accurately) that -48 and 24 are eigenvalues of T. Without using a computer or writing anything down, find the third eigenvalue of T.

SOLUTION: The sum of the eigenvalues of T equals the sum of the diagonal terms of the matrix above (both quantities equal $\operatorname{trace} T$). The sum of the diagonal terms of the matrix above equals 12. The sum of two of the eigenvalues of T, -48 and 24, equals -24. Because the sum of all three eigenvalues of T must equal 12, the third eigenvalue of T must be 36.

~~13.~~ **14.** Prove or give a counterexample: if $T \in \mathcal{L}(V)$ and $c \in \mathbf{F}$, then $\operatorname{trace}(cT) = c \operatorname{trace} T$.

SOLUTION: Suppose $T \in \mathcal{L}(V)$ and $c \in \mathbf{F}$. To prove that $\operatorname{trace}(cT) = c \operatorname{trace} T$, consider a basis of V. Then $\operatorname{trace} T$ equals the sum of the diagonal terms of the matrix of T with respect to this basis. The matrix of cT, with respect to the same basis, equals c times the matrix of T. Thus the sum of the diagonal terms of the matrix of cT equals c times the sum of the diagonal terms of the matrix of T. In other words, $\operatorname{trace}(cT) = c \operatorname{trace} T$.

~~14.~~ **16.** Prove or give a counterexample: if $S, T \in \mathcal{L}(V)$, then

$$\operatorname{trace}(ST) = (\operatorname{trace} S)(\operatorname{trace} T).$$

SOLUTION: Define $S, T \in \mathcal{L}(\mathbf{F}^2)$ by $S(x, y) = T(x, y) = (-y, x)$. Then with respect to the standard bases the matrix of S (which of course equals the matrix of T) is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Thus $\operatorname{trace} S = \operatorname{trace} T = 0$. However, $ST = -I$, so $\operatorname{trace} ST = -2$. Thus for this choice of S and T, we have $\operatorname{trace}(ST) \neq (\operatorname{trace} S)(\operatorname{trace} T)$.

Of course there are also many other examples.

~~15.~~ **17.** Suppose $T \in \mathcal{L}(V)$. Prove that if $\operatorname{trace}(ST) = 0$ for all $S \in \mathcal{L}(V)$, then $T = 0$.

SOLUTION: Suppose that $\operatorname{trace}(ST) = 0$ for all $S \in \mathcal{L}(V)$. Then $\operatorname{trace}(TS) = 0$ for all $S \in \mathcal{L}(V)$ (by 10.12). Suppose that there exists $v \in V$ such that $Tv \neq 0$. Then (Tv) can be extended to a basis $(Tv, u_1, \dots, u_n)$ of V. Define $S \in \mathcal{L}(V)$ by

$$S(aTv + b_1u_1 + \dots + b_nu_n) = av.$$

Thus $S(Tv) = v$ and $Su_j = 0$ for each j. Hence $(TS)(Tv) = T(S(Tv)) = Tv$ and $(TS)(u_j) = 0$ for each j. This implies that with respect to the basis $(Tv, u_1, \dots, u_n)$, the matrix of TS consists of all 0's except for a 1 in the upper-left corner. Thus $\operatorname{trace}(TS) = 1$. This contradiction shows that our assumption that $Tv \neq 0$ must have been false. Thus $Tv = 0$ for every $v \in V$, which means that $T = 0$.

~~16.~~ **18.** Suppose V is an inner-product space and $T \in \mathcal{L}(V)$. Prove that if $(e_1, \dots, e_n)$ is an orthonormal basis of V, then

$$\operatorname{trace}(T^*T) = \|Te_1\|^2 + \dots + \|Te_n\|^2.$$

Conclude that the right side of the equation above is independent of which orthonormal basis $(e_1, \dots, e_n)$ is chosen for V.

SOLUTION: Suppose that $(e_1, \dots, e_n)$ is an orthonormal basis of V. Then

$$\begin{aligned} \operatorname{trace} T^*T &= \langle T^*Te_1, e_1\rangle + \dots + \langle T^*Te_n, e_n\rangle \\ &= \langle Te_1, Te_1\rangle + \dots + \langle Te_n, Te_n\rangle \\ &= \|Te_1\|^2 + \dots + \|Te_n\|^2. \end{aligned}$$

Because $\operatorname{trace} T^*T$ does not depend upon the choice of a basis of V, the formula above shows that $\|Te_1\|^2 + \dots + \|Te_n\|^2$ is independent of the orthonormal basis $(e_1, \dots, e_n)$.