Solutions for Additional Exercises in `exercises5.pdf`

5.A.X1. We shall use the 2×2 determinant test for eigenvalues and then find the eigenvectors.

For the first matrix, the polynomial $\det A - tI$ is $5 - 6t + t^2 = (t-1)(t-5)$, so the eigenvalues are 1 and 5. To compute the eigenvectors, we must find the null spaces of the following matrices:

$$A - I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \quad , \quad A - 5I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

In the first case the null space is spanned by the column vector corresponding to $(1, -1)$, while in the second the null space is spanned by the column vector corresponding to $(1, 1)$.■

For the second matrix, the polynomial is $\det A - tI = 5 - 6t + t^2 = (t-1)(t-5)$, so the eigenvalues are again 1 and 5. To compute the eigenvectors, we must find the null spaces of the following matrices:

$$A - I = \begin{pmatrix} -1 & -5 \\ 1 & 5 \end{pmatrix} \quad , \quad A - 5I = \begin{pmatrix} -5 & -5 \\ 1 & 1 \end{pmatrix}$$

In the first case the null space is spanned by the column vector corresponding to $(5, -1)$, while in the second the null space is spanned by the column vector corresponding to $(1, -1)$.■

For the third matrix, the polynomial is $\det A - tI = 18 - 9t + t^2 = (t-3)(t-6)$, so the eigenvalues are 3 and 6. To compute the eigenvectors, we must find the null spaces of the following matrices:

$$A - 3 = \begin{pmatrix} 5 & 5 \\ -2 & 2 \end{pmatrix} \quad , \quad A - 6I = \begin{pmatrix} 2 & 5 \\ -2 & -5 \end{pmatrix}$$

In the first case the null space is spanned by the column vector corresponding to $(1, -1)$, while in the second the null space is spanned by the column vector corresponding to $(5, -2)$.■

5.A.X2. The polynomial whose roots are eigenvalues is $1 - 2\cos\theta + t^2$. By the Quadratic Formula, this polynomial has no real roots if and only if $\cos^2\theta - 1 < 0$, or equivalently $\cos\theta \neq \pm 1$. The latter is true precisely when θ is not an integral multiple of π.■

5.B.X1. Follow the hint, letting W_k be the subspace spanned by the first $k \geq 0$ unit vectors; by convention $W_0 = \{0\}$. Then the linear transformation $X \to AX$ sends the k^{rmth} unit vector into W_{k-1} by our assumptions. Now if $Y \in W_k$, then Y is a linear combination of the first k unit vectors, so it follows that $X \to AX$ sends Y into W_{k-1}. Similarly, A sends $AX \in W_{k-1}$ to $A^2X \in W_{k-1}$, and more generally we have that $A^jX \in W_{k-j}$. Since W^n is the whole vector space, it follows that for every X the vector A^nX lies in $W_0 = \{0\}$. Finally, since a matrix B is zero if for all column vectors Y (with the right number of rows!) we have $BY = 0$, it follows that $A^n = 0$.■

5.B.X2. It is helpful to write out the conditions for a matrix to be unitriangular; namely P is (upper) unitriangular if $p_{i,j} = 0$ for $i > j$ and $p_{i,i} = 1$ for all i.

Suppose now that A and B are unitriangular, and let $C = AB$. Consider an entry $c_{i,j}$ where $i > j$ and expand it as usual:

$$c_{i,j} \quad = \quad \sum_k a_{i,k} b_{k,j}$$

Since B is triangular, it follows that all terms $b_{k,j}$with $k > j$ are zero. For the remaining terms, we have $k \leq j < i$ and therefore $a_{i,k} = 0$ in these cases too. Therefore each of the monomials in the formula for $c_{i,j}$ has a factor equal to zero, and therefore the sum, which is $c_{i,j}$, must also be zero.

Now consider $c_{i,i}$. The preceding discussion shows that the summands for which $k > i$ are all zero and that the same is true for all summands such that $k < i$. The only remaining case is when $k = i$. In this case $a_{i,k} = 1 = b_{k,i}$, so the net contribution of this term to the summand is 1. Since all other terms in the summation are zero, it follows that $c_{i,i} = 1$ for all i.■

Generalization. For more general (upper) triangular matrices, the preceding considerations show that $c_{i,j} = 0$ if $i > j$ and $c_{i,i} = a_{i,i}b_{i,i}$.■

5.C.X1. The preceding generalization shows that the diagonal entries of the power matrix A^k are the k^{rmth} powers of the diagonal entries for A. Also, direct calculation shows that if P and Q are upper triangular then so is $P + Q$ (the matrices must have the same size) and the diagonal entries are the sums of the diagonal entries for P and Q. Combine these to obtain the assertion about the diagonal entries of the matrix $p(A)$.■

5.C.X2. We shall use the hint. Let v_j be an eigenvector for the eigenvalue $a_{j,j}$; recall that the distinct eigenvalues of A are the diagonal entries $a_{j,j}$.

Since multiplication of polynomials is commutative, we can write $p(t)$ as a product $q_j(t) \cdot (t - a_{j,j})$, where q_j is a product of all the linear factors except $t - a_{j,j}$. It then follows that

$$p(A)v_j \quad = \quad q_j(A)\,(A - a_{j,j})v_j \quad = \quad q_j(A)0 \quad = \quad 0\ .$$

Since j was arbitrary, it follows that $p(A)x = 0$ for all x in a basis for the space of column vectors and therefore $p(A)$ is the zero matrix.■

Here is an example to illustrate the strength of the conclusion in the preceding exercise. If A is any matrix of the form

$$\begin{pmatrix} 1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 2 & b_3 & b_4 & b_5 \\ 0 & 0 & 3 & c_4 & c_5 \\ 0 & 0 & 0 & 4 & d_5 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

and $p(t) = (t-1)(t-2)(t-3)(t-4)(t-5)$, then we automatically know that $p(A) = 0$ no matter what coefficients lie above the main diagonal.