Solutions to Exercises 6

6A. Inner products

<u>X1.</u> This amounts to finding solutions to the linear homogeneous system

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

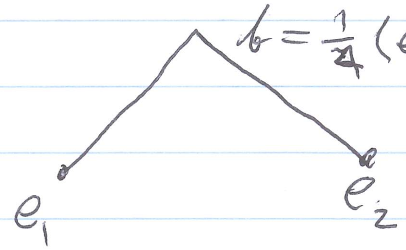
This system is equivalent to the system with matrix $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} = 0$, which has a 1-dim solution space spanned by $(0, -1, 1, 0)$.

<u>X2.</u> Need to solve $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

This system is equivalent to the one associated to $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. For the latter, we can let x_3 & x_4 vary arbitrarily, and then x_1 & x_2 are forced.

Basis: $(1, -2, 1, 0)$
$(2, -3, 0, 1)$

X3.



$$e_1 - b = \frac{3}{4}e_1 - \frac{1}{4}(e_2 + e_3 + e_4)$$
$$e_2 - b = \frac{3}{4}e_2 - \frac{1}{4}(e_1 + e_3 + e_4)$$

$$\cos\theta = \frac{\langle e_1 - b, e_2 - b\rangle}{|e_1 - b||e_2 - b|}$$

$$|e_1 - b| = \sqrt{\frac{9}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$= \frac{(e_1 - b)(e_2 - b)}{\frac{3}{4}} = \frac{-\frac{1}{4}}{\frac{3}{4}} = -\frac{1}{3} \text{ so}$$

$\theta \doteq 109.47122\ldots$ degrees.

6.B. Orthogonality and dimension

X1. (a) Let $b = (2,2,3)$, $a = (1,1,0)$. $b = b_1 + b_2$ where

$$b_1 = \frac{b \cdot a}{a \cdot a} a = \frac{4}{2}(1,1,0) = (2,2,0)$$

Hence $b_2 = (0,0,3)$.

(b) Here $a = (1,1,1)$, $b = (1,-1,1)$. In this case

$$b_1 = \frac{b \cdot a}{a \cdot a} a = \frac{1}{3} a = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \text{ and}$$

$$b_2 = \left(+\frac{2}{3}, -\frac{4}{3}, +\frac{2}{3}\right).$$

X2. (a) Call the three vectors v_1, v_2, v_3.

Then $w_1 = v_1 = (0,0,1,1)$,

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = (0,1,1,0) - \frac{1}{2}(0,0,1,1)$$

$$= \left(0, 1, \frac{1}{2}, -\frac{1}{2}\right),$$

$$\cancel{w_3 = v_3 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 =}$$

Since $v_3 = v_1$, the vectors w_1 & w_2 span

(b) In this case

$$w_1 = v_1 = (1, 1, 1, 1),$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} = (-1, 4, 4, 1) - \overset{2}{\cancel{\frac{8}{4}}}(1,1,1,1)$$

$$= (-3, 2, 2, -1),$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 =$$

$$(4, -2, -2, 0) - \overset{0}{\cancel{\frac{0}{4}}}(1,1,1,1) - \underset{-10/9}{\cancel{\frac{-20}{18}}}(-3, 2, 2, -1)$$

$$= \left(4 - \frac{10}{3}, -2 + \frac{20}{9}, -2 + \frac{20}{9}, -\frac{10}{9}\right) =$$

$$\left(\frac{2}{3}, \frac{2}{9}, \frac{2}{9}, -\frac{10}{9}\right).$$ Note that

$$9w_3 = (6, 2, 2, -10).$$

When one does computations, it is always a good idea to check the results; i.e., check that $\left.\begin{array}{c}\langle w_1, w_2\rangle \\ \langle w_1, w_3\rangle \\ \langle w_2, w_3\rangle\end{array}\right\}$ are all zero.

6C. Orthogonal complements, etc.

X1 Prove that $U^\perp \cap W^\perp = (U+W)^\perp$.
(correcting misprint)

① Show $U^\perp \cap W^\perp \subseteq (U+W)^\perp$. Let $x \in U^\perp \cap W^\perp$. Then $\langle x, u\rangle = 0 = \langle x, w\rangle$ for all $u \in U$ & $w \in W$. If $y \in U+W$, write $y = u+w$ with $u \in U$, $w \in W$. Then $\langle x, y\rangle = \langle x, u+w\rangle = \langle x, u\rangle + \langle x, w\rangle = 0+0 = 0$. Hence $x \in (U+W)^\perp$.

② Reverse inclusion: Since $U, W \subseteq U+W$ we have $(U+W)^\perp \subseteq U^\perp, W^\perp$, which means $(U+W)^\perp \subseteq U^\perp \cap W^\perp$.

X2. We know that $E^2 = E$. Therefore if $T = (I - 2E)^3$, then $T^2 = (I-2E)^2 =$

$I - 2(2E) + 2E^2 = I - 4E + 4E = I$.

Also $x \in W \Rightarrow Ex = x \Rightarrow Tx = (I-2E)x$
$= x - 2Ex = x - 2x = -x$. Hence x is an eigenvector for T with eigenvalue -1.

X3. First find an orthogonal basis for W if v_1 + v_2 are given as in the problem.

Then $w_1 = v_1 = (1,1,0)$ and

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = (0,1,1) - \frac{1}{2}(1,1,0)$$

$$= \left(\frac{1}{2}, -\frac{1}{2}, 1\right).$$

So an orthogonal basis for W is given by $(1,1,0) = x_1$ and $(1,-1,2) = x_2$.

It follows that

$$E e_j = e_j - \frac{\langle e_j, x_1\rangle}{\langle x_1, x_1\rangle} x_1 - \frac{\langle e_j, x_2\rangle}{\langle x_2, x_2\rangle} x_2$$

Now substitute e_1, e_2, e_3 for e_j.

$$E e_1 = (1,0,0) - \frac{1}{2}(1,1,0) - \frac{1}{6}(1,-1,2) =$$

$$\left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$

$$E e_2 = (0,1,0) - \frac{1}{2}(1,1,0) - \frac{(-1)}{6}(1,-1,2) =$$

$$\left(-\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

Finally,

$$E e_3 = (0,0,1) - \frac{0}{2}(1,1,0) - \frac{2}{6}(1,-1,2) =$$

$$\left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

So the matrix of E is given by

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

One way to partially check this is to compute E^2 and see if it equals E. If not, there is a mistake in the solution. Also, it turns out that $E = {}^{T}E$ (transpose) must hold.