Solutions for aabUpdate11.132.w17.pdf

3. First show $V_+ = \text{Image}\,\frac{1}{2}(T+I)$
$V_- = \text{Image}\,\frac{1}{2}(T-I)$. (given $T^2 = I$).

$\underline{\text{Image}\,\frac{1}{2}(I+T) \subseteq V_+}$ $\quad T\left(\frac{1}{2}(x+Tx)\right) = \frac{1}{2}(Tx + \underbrace{T^2}_{=I}x)$
$= \frac{1}{2}(x+Tx)$

$\underline{V_+ \subseteq \text{Image}\,\frac{1}{2}(I+T)}$. $\quad Tx = x \Rightarrow \frac{1}{2}(I+T)x =$
$\frac{1}{2}x + \frac{1}{2}x = x$,
so $x = \frac{1}{2}(I+T)x$.

$\underline{\text{Image}\,\frac{1}{2}(T-I) \subseteq V_-}$. $\quad T\left(\frac{1}{2}(T-I)x\right) =$
$\frac{1}{2}(T^2x - Tx) = \frac{1}{2}(I-T)x =$
$-\frac{1}{2}(T-I)x$.

$\underline{V_- \subseteq \text{Image}\,\frac{1}{2}(T-I)}$ $\quad Tx = -x \Rightarrow \frac{1}{2}(T-I)(-x)$
$= -\frac{1}{2}(Tx - x) = -\frac{1}{2}(-x-x) = -\frac{1}{2}(-2x) = x$.

Next, $\mathbb{R}^n = V_+ \oplus V_-$ and $V_+ \cap V_- = \{0\}$

true since these are eigenspaces for different eigenvalues

$x = \frac{1}{2}(T+I)x - \frac{1}{2}(T-I)x \in V_+ + V_-$.

So $Tx = x$ on V_+ and $Tx = -x$ on V_-. If we take bases $\mathcal{B}_+$ and $\mathcal{B}_-$ for V_+ and V_- respectively, then T is diagonalizable with respect to β_+ and β_-, and the trace of the matrix for T with respect to $\beta_+ \cup \beta_-$ is equal to $(\#1\text{'s}) - (\# -1\text{'s}) = \dim V_+ - \dim V_-$. ∎

4. Write $A = \lambda I + N$ where N is the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $A^3 = (\lambda I + N)^3 = \lambda^3 I + 3\lambda^2 N + 3\lambda N^2 + N^3 =$

We can do this since λI + N commute!

$$\begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda & 1 \\ 0 & \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{bmatrix}$$

This is not in Jordan form since the $(1,3)$ $(1,4)$ and $(2,4)$ entries are nonzero. To find the Jordan form, look at $(\lambda I + N)^3 - \lambda^3 I =$

$$\begin{bmatrix} 0 & 3\lambda^2 & 3\lambda & 1 \\ 0 & 0 & 3\lambda^2 & 3\lambda \\ 0 & 0 & 0 & 3\lambda^2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

Then rank $B = 3$, so its Jordan form must be

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

therefore the Jordan form of A^3 is

$$\begin{bmatrix} \lambda^3 & 1 & 0 & 0 \\ 0 & \lambda^3 & 1 & 0 \\ 0 & 0 & \lambda^3 & 1 \\ 0 & 0 & 0 & \lambda^3 \end{bmatrix}. \blacksquare$$

5. $\det cA = c^n \det A$ if A is $n \times n$. If $c = -1$ and n is even this means that $\det(-A) = \det A$. $\blacksquare$

6.
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 0 \\ 0 & 7 & 8 & 1 & 0 \\ 0 & 9 & 1 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix} \xrightarrow[\text{others}]{\text{Subtr. mults of 1st from}} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 8 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 5 & 0 & 3 & 2 & 1 \end{bmatrix}$$

$\Delta = 1$

$$\xrightarrow{\text{Same for 2nd row}} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 5 & 4 & 0 & 2 & 1 \end{bmatrix}$$

$\Delta = 1$

Two rows are equal, so $\det = 0$. $\blacksquare$

7.
$$\begin{vmatrix} 3-t & 1 & 0 \\ 0 & 1-t & 0 \\ 4 & 2 & 1-t \end{vmatrix} = (3-t)(1-2t+t^2)$$

only one of the six terms is nonzero!

$$\begin{vmatrix} 2-t & 0 & 0 \\ -2 & -2-t & 2 \\ 5 & -10 & 7-t \end{vmatrix} = (t^2-4)(7-t) + 20 = -t^3 + 5t^2 + 4t - 8.$$

The polynomial should read

$$\boxed{a_n z^n + \dots + a_1 z + a_0}$$

and for the given matrix the characteristic polynomial is

$$\chi(t) = -t^3 + 6t^2 - 5t + 1.$$

By Gauss' result, if a rational number $\frac{p}{q}$ (p, q integers, $q \neq 0$, reduced to least terms) is a root, then p divides $+1$ and q divides -1. Hence the only possible rational roots are ± 1. But $\chi(1) = 1$ and $\chi(-1) = 13$, so neither of these is a root. Hence there are no rational roots.

9. W_k = all v such that $Av = kv$ is 1-dim. for $k = 2$ and 3. If V_2 + V_3 are the corresponding V_λ's we know $\dim V_2 + \dim V_3 = 4$. Now list the possibilities, letting $n(\lambda) = \dim V_\lambda$.

$n(2)=1,\ n(3)=3$

$$\left[\begin{array}{c|ccc} 2 & 0 & 0 & 0 \\ \hline 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$n(2)=n(3)=2$

$$\left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$n(2)=3,\ n(3)=1$ First matrix with the roles of 2 and 3 switched.

10. Minimal polynomial $= z^2 \Rightarrow \dim V_0 = 4$ and there is at least one 2×2 Jordan block, but there is no 3×3 Jordan block. Here are the possibilities

$$\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$