Review of basic linear algebra

vectors scalars

A system of scalars is an object with two binary operations ($+$ and $\times$ or $\cdot$) satisfying standard rules for $+, -, \times, \div$; provided the denominator is nonzero in the last case.

Examples Real numbers $\mathbb{R}$

Complex numbers $\mathbb{C}$ = all $z = x + yi$ with $x, y \in \mathbb{R}$ and $i^2 = -1$.

Probably the least elementary feature for $\mathbb{C}$ is division by a nonzero complex number $a + bi$ where a & b are not both zero. But this follows from a simple identity:

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$$

(note the denominator on the right is positive if a & b are not both 0)

If $z = x + yi$, its conjugate $\bar{z} = x - yi$.

The standard examples of vectors are elements of $\mathbb{R}^n$. The n coords. give the magnitude and direction of the vector.

Vector Spaces over Scalars

HAS Notions of vector addition & scalar multiplication which satisfy the usual algebraic laws for vectors in $\mathbb{R}^n$. All of this makes sense, and extends, to $\mathbb{C}^n$. At this point we ignore structures such as dot products and cross products. In particular, everything also generalizes to scalars in an arbitrary field (4 arithmetic ops as above), so that F^n is a vector space over F if F is a field.

Other examples

Continuous real valued functions on the interval $[0,1]$: Set $f+g(t) := f(t)+g(t)$, $[cf](t) := c[f(t)]$.

Also, bounded functions, differentiable functions, etc.

SUBSPACES V vector space over F. Then a nonempty subset $W \subseteq V$ is a subspace if it is a vector space in its own right with respect to vector addition & scalar multiplication (equivalently, W is closed under taking vector sums and scalar products).

Examples $W \subseteq F^n$ defined by an equation $\sum_j a_j x_j = 0$, where not all a_j's are zero.

(Lines through origin in $\mathbb{R}^2$, planes through origin in $\mathbb{R}^3$)

V = continuous functions, W = differentiable functions.

$\{0\} \subseteq V$ is always a subspace (zero subspace).

Theorem (i) If W_1 and W_2 are subspaces of V, then so is $W_1 \cap W_2$.

(ii) If W_1 and W_2 are subspaces of V, then so is $W_1 + W_2$ = all $x+y$ where $x \in W_1$, $y \in W_1$.

Note If the union $W_1 \cup W_2$ is a subspace, then either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Internal direct sums. Write $V \approx W_1 \oplus W_2$ if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

Standard external direct sum $W_1 \oplus W_2 = W_1 \times W_2$, coordinatewise addition & scalar multiplication. Then
$$W_1 \oplus W_2 \approx (W_1 \times \{0\}) \oplus (\{0\} \times W_2).$$

II - Bases and dimensions.

Central issue: For a nonnegative integer n, give an intrinsic description of n-dimensionality.

Linear combinations of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ = finite sums $\sum c_{\alpha(i)} \mathbf{v}_{\alpha(i)}$, where each $c_{\alpha(i)}$ is a scalar.

$S \neq \emptyset$ If S is a set of vectors, the span of S consists of all linear combinations of vectors in S.

If S is a set of vectors, then S is linearly independent if every vector has at most one expansion as a linear combination of vectors in S.

Say S is <u>linearly dependent</u> if this does not hold.

Examples in $\mathbb{R}^3$

S	spans?	lin indep.?
$\{(1,0,0),(1,1,0)\}$	no (can't express $(0,0,1)$).	yes
$\{(1,0,0),(1,1,0),(0,0,1)\}$	yes	yes
$\{(1,0,0),(1,1,0),(0,1,1),(0,0,1)\}$	yes	no ($a-b+c-d=0=0a+0b+0c+0d$)

<u>Alternative characterization</u> S is lin indep $\iff$ whenever $\sum c_{\alpha(i)} v_{\alpha(i)} = 0$, we must have $c_{\alpha(i)} = 0$ for all i.

<u>Proposition</u>
If $S \subseteq T$ and S spans, then so does T.
If $T \subseteq S$ and S is lin indep., then so is T.

A <u>basis</u> is a set S which is linearly independent and spans.

<u>Standard unit vector basis of F^n:</u>

$(1,0,\ldots,0) = e_1$, $e_i = (0,\ldots,0,\overset{i}{1},0\ldots0)$, $e_n = (0,\ldots,0,1)$.

FUNDAMENTAL RELATIONSHIP

If $A \subseteq V$ is a set of p lin indep vectors and $B \subseteq V$ is a set of q spanning vectors, then $p \leq q$. (p, q finite)

Example The space of cont fcns. has no finite spanning set.

COROLLARY 1 If V has finite bases A and B, then these two subsets have the same number of elements. (This number is called the dimension of V, written $\dim V$)

Theorem Suppose V is finite dimensional.

(i) If S spans V, then S contains a basis.

(ii) If S is linearly independent, then there is a basis T of V such that $S \subseteq T$.

Dimension formulas.

(i) If W is a subspace of V and $\dim V = n$, then W is finite dimensional and $\dim W \leq \dim V$. Equality holds $\Longleftrightarrow W = V$.

(iii) If W_1 & W_2 are subspaces of V and $\dim V = n$, then we have

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

<u>One geometric formulation</u>

$V = \mathbb{R}^3$ & $\dim W_1 = \dim W_2 = 2$.

Then $\dim W_1 \cap W_2 = 1$. (Two planes with a point in common also have a line in common).

<u>Note</u> If V does not have a finite basis we say that V is infinite-dimensional & sometimes write $\dim V = \infty$.

<u>Useful recognition principle</u> Suppose that S is a finite basis for the subspace $W \subseteq V$ and $y \in V$ but $y \notin W$. Then $S \cup \{y\}$ is linearly independent.