Proof of the "Simple recognition priniple"

Here is a proof of the final result stated in `firstcourse1.pdf`:

THEOREM. *Suppose that S is a finite basis for the vector subspace $W \subset V$ (where V is also a vector space) and $y \in V$ is not in W. Then the set $S \cup \{y\}$ is linearly independent.*

Proof. Suppose that we have a nontrivial expression

$$\sum_{s \in S} c_s\, s \;+\; dy \quad = \quad 0$$

where not all the coefficients c_s and d are zero. If $d = 0$ this means that $\sum_s \; c_s\, s = 0$ where some $c_s \neq 0$. Since the set S is linearly independent, it follows that each $c_s = 0$, a contradiction. Therefore $d \neq 0$ and hence we have

$$y \quad = \quad \sum_s -d^{-1} c_s\, s$$

which means that $y \in W$. Since the latter is assumed to be false, we have a contradiction. Therefore the set $S \cup \{y\}$ must be linearly independent.∎