Two vectors

This is a review of some concepts from a first linear algebra course which discusses their meanings for pairs of vectors and 2×2 matrices.

Linear independence One characterization of linear independence is that no vector in the set is a linear combination of the others (special considerations apply to $\{0\}$). If we have a set of two vectors, this is false $\Longleftrightarrow$ one vector is a scalar multiple of the other.

Consequence Two vectors form a basis for $\mathbb{R}^2 \Longleftrightarrow$ neither is a scalar multiple of the other.

Proof. A theorem says that a set of n vectors in $\mathbb{R}^n$ is a basis if it is known to be linearly independent or known to span. If $n = 2$, we may apply the preceding discussion.

Invertible 2x2 matrices

By the preceding, a 2×2 matrix is invertible $\Leftrightarrow$ its columns are linearly independent, so it is not invertible $\Leftrightarrow$ one column is a multiple of the other.

CLAIM $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is not invertible $\Leftrightarrow$

$ad - bc = 0$.

Proof. If either column has only zeros, then the matrix is not invertible since $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} x \\ y \end{pmatrix}$ for all x, y, and $ad - bc = 0$ in this case. So suppose neither column is zero.

Noninvertibility implies either

$\begin{pmatrix} a \\ b \end{pmatrix} = p \begin{pmatrix} c \\ d \end{pmatrix}$ for some p, or

$\begin{pmatrix} c \\ d \end{pmatrix} = q \begin{pmatrix} a \\ b \end{pmatrix}$ for some q. In these

cases we have

$$ad - bc = (pc)d - (pd)c = 0$$

and $ad - bc = a(qb) - (qa)b = 0$

respectively.

Conversely, suppose $ad - bc \neq 0$. Now

$\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a \neq 0$ or $b \neq 0$.

Suppose $a \neq 0$, and also ~~suppose~~

write $c = ka$. Then $d \neq \frac{bka}{a} = bk$,

so $\begin{pmatrix} c \\ d \end{pmatrix}$ is not a multiple of $\begin{pmatrix} a \\ b \end{pmatrix}$.

Can $\begin{pmatrix} a \\ b \end{pmatrix} = k \begin{pmatrix} c \\ d \end{pmatrix}$ for some k? We claim

it can't. If it did, then $0 \neq a = kc \Rightarrow$

$k \neq 0$, so that $\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{k} \begin{pmatrix} a \\ b \end{pmatrix}$, and

we have shown this is impossible.

If $d \neq 0$ we can proceed similarly switching the roles of the first and second coordinates. $\blacksquare$