6B. Orthogonality and dimension

One way of characterizing 2- and 3-dimensional vectorspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$ is that there are 2 or 3 (resp) distinct and mutually perpendicular directions. One aim of this section is to generalize this intuitive notion to higher dimensions.

Prop. If $\{v_1, \ldots, v_m\}$ is a set of nonzero, mutually perpendicular vectors in an inner product space V, then $\{v_1, \ldots, v_m\}$ is linearly independent.

PROOF. Suppose $\sum c_j v_j = 0$. For each k, we then have $0 = \langle \sum c_j v_j, v_k \rangle = \sum c_j \langle v_j, v_k \rangle$. Since $\langle v_j, v_k \rangle = 0$ if $j \neq k$, this yields $c_k |v_k|^2 = 0$, and since $v_k \neq 0$ we have $|v_k|^2 > 0$ and hence $c_k = 0$. Therefore the given set of vectors is lin. indep. ∎

Hence $\dim V = n \Rightarrow$ every set of nonzero mutually $\perp$ vectors has $\leq n$ elements.

Question Is there such a set with exactly $n = \dim V$ elements?

Example $V = \mathbb{R}^n$ or $\mathbb{C}^n$, e_j = unit vector with jth coord $= 1$, all others equal to zero. Then $\{e_1, \dots, e_n\}$ is mutually orthogonal.

Orthonormal set. $\{v_1, v_2, \dots, v_n\}$ is orthonormal if the vectors are mutually $\perp$ and $|v_j| = 1$ for all j.

Gram-Schmidt Orthonormalization Process

Let V be an ~~vector~~ inner product space with basis $\{v_1, \dots, v_n\}$. Then V has an orthonormal basis $\{w_1, \dots, w_n\}$ such that for each k we have

$$\mathrm{Span}\{w_1, \dots, w_k\} = \mathrm{Span}\{v_1, \dots, v_k\}.$$

PROOF Note that

1. This applies to finite dim subspaces of inner product spaces (restrict the inner product).
2. The conclusion suggest that $\{w_1, \dots, w_n\}$ is constructed recursively, and it is.

$k=1$ $v_1 \neq 0$; let $u_1 = \frac{1}{|v_1|} v_1$.

Suppose true for $k < n$, $k \geq 1$

span same subspace

Find a vector $w \perp \{v_1, \dots, v_k\}$, $\{u_1, \dots, u_k\}$

such that w is a lin. comb. of $v_1, v_2, \dots, v_k, v_{k+1}$:

$$w = v_{k+1} - \sum_{j=1}^{k} \langle w, u_j \rangle u_j.$$

This vector is $\perp u_1, \dots, u_k$. Now if w is perpendicular to $a_1, \dots, a_k$ then it is also $\perp$ to every linear combination of $a_1, \dots, a_k$ because

$$\langle w, \sum y_j a_j \rangle = \sum \overline{y_j} \underbrace{\langle w, a_j \rangle}_{=0} = 0.$$

Since nonzero orthogonal sets are linearly indep, w isn't a lin. comb. of $v_1, \dots, v_k$ (otherwise v_{k+1} would be). In particular, $w \perp u_j$ and $w \neq 0$ & $\mathrm{Span}\{u_1, \dots, u_k, w\} = \mathrm{Span}\{v_1, \dots, v_{k+1}\}$.

Finally, let $u_{k+1} = \frac{1}{|w|} w$; then $\{u_1, \dots, u_{k+1}\}$ is orthonormal with the same span as $\{v_1, \dots, v_{k+1}\}$.

The G.-S. process provides an effective algorithm for converting an arbitrary basis of $\mathbb{R}^n$ to an orthonormal one, and likewise for all subspaces of $\mathbb{R}^n$. It would have been nice if the text book had given a worked example. See gram-schmidt.pdf for an example from Schaum's Outline of Linear Algebra.

A FEW USEFUL IDENTITIES

Let $\{w_1, \dots, w_n\}$ be an orthonormal basis for V.

1. If $x \in V$ and $x = \sum c_j w_j$, then
   $c_j = \langle x, w_j \rangle$ (Formula 6.30, p 182)

2. If $x = \sum a_j w_j$ and $y = \sum b_j w_j$, then

   $\langle x, y \rangle = \sum a_j \overline{b_j}$.

Also look at Thm. 6.38 (p.186), Example 6.29 (p. 181), Thm. 6.35 (p. 185)